AN ELEMENTARY THEORY OF COMPLEX NUMBERS

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1 COMPLEX NUMBERS

Before we develop the theory of complex numbers, we first review the algebraic laws of addition, subtraction, multiplication and division of real numbers. In addition we will assume familiarity with negative numbers, rational numbers, irrational numbers and transcedental number.

We now list some of the properties of real numbers. Let $a, b, c, \ldots; x, y, z, \ldots$ be a set of real numbers. On these real numbers we can perform the following operations:

1. $a + b = b + a$. Law of addition is *COMMUTATIVE.*
2. $ab = ba$. Law of multiplication is *COMMUTATIVE.*
3. $a + (b + c) = (a + b) + c$. Law of addition is *ASSOCIATIVE.*
4. $(ab)c = a(bc)$. Law of multiplication is *ASSOCIATIVE.*
5. $(a + b)c = ac + bc$. The law of multiplication is *DISTRIBUTIVE* over addition.
6. Given two real numbers a and b such that
$$x + a = b,$$
there exists a unique solution
$$x = (b - a).$$
7. Given two real numbers a and b such that
$$ax = b,$$
there exists a unique solution
$$x = b/a,$$
provided $a \neq 0$.

Additionally, we also assume that:

1. There exists a number 0 *(zero)* such that $a + 0 = a$.
2. For every number a there exists a number b such that $a + b = 0$.
3. There exists a number $1 \neq 0$ such that $1 \cdot a = a$ for every a.
4. For every number $a \neq 0$, there exists a number b called the *reciprocal* number, such that $a \cdot b = 1$.

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2 ABSOLUTE VALUE

The absolute value or *magnitude* of a real number a, which may be a number on the positive real axis or the negative real axis is written as $|a|$. Thus $|a| = a$ and $|-a| = a$.

Absolute values of real numbers obey the following two rules:

1. $|a+b| \leq |a| + |b|$.

   Example 1. Let $a = 3$, $b = -5$. Then,

   $$a + b = 3 - 5 = -2,$$

   $$|a+b| = |-2| = 2,$$

   $$|a| + |b| = |3| + |-5| = 3 + 5 = 8.$$

   Hence, $|a+b| \leq |a| + |b|$. $\diamond$

2. $|ab| = |a||b|$, i.e., absolute value of the product of two real numbers is equal to the product of the absolute values.

   Example 2. Let $a = -3$, $b = 5$. Then,

   $$\begin{aligned} ab &= -3 \times 5 = -15, \\ |ab| &= |-15| = 15, \\ |a||b| &= |-3| \times |5| = 3 \times 5 = 15. \end{aligned}$$

   Hence, $|ab| = |a||b|$. $\diamond$

3 GEOMETRY OF COMPLEX NUMBERS

We denote real numbers on the x-axis, imaginary numbers on the y-axis and complex numbers $z = x + iy$ by points (x, y) in the rectangular plane. We often refer to the point (x, y) as the "point z", the x-axis is often called *the real axis*, y-axis is often called *the imaginary axis.*

The distance between the point z and the origin is called the absolute value or *modulus* of z. It is denoted by $|z|$, where

$$|z| = r = \sqrt{x^2 + y^2}. \tag{1}$$

Modulus is a non-negative real number and therefore we always use positive sign before the radical.

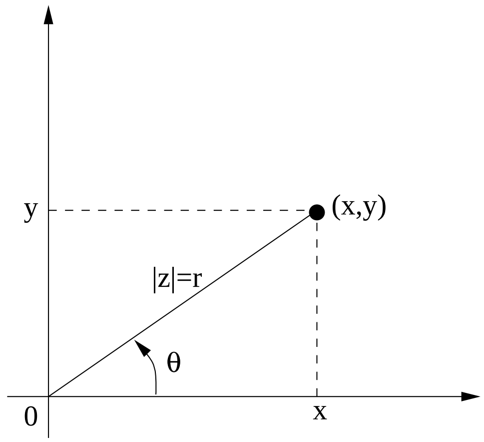

Figure 1.

Example 1. Let $z_1 = 3 + 4i$. Find the absolute value.
The absolute value is

$$\begin{aligned} |z_1| &= \sqrt{3^2 + 4^2}\,, \\ &= \sqrt{9 + 16}\,, \\ &= 5. \qquad \diamond \end{aligned} \tag{2}$$

Consider another point $z_2 = 4 + 5i$. In this case the absolute value is

$$\begin{aligned} |z_2| &= \sqrt{4^2 + 5^2}\,, \\ &= \sqrt{16 + 25}\,, \\ &= \sqrt{41} = 6.4031242 \cdots. \qquad \diamond \end{aligned} \tag{3}$$

Comparing the absolute value of z_1 with z_2, we find $|z_1| < |z_2|$ and therefore point z_1 is closer to the origin than point z_2.

We can now determine the distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. To find the distance between these two points, we first find the difference

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{4}$$

The absolute value of the difference is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{5}$$

In Example 1, $z_1 - z_2 = -1 - i$, and therefore the distance between the two points is

$$|z_1 - z_2| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}. \qquad \diamond \tag{6}$$

4 POLAR REPRESENTATION OF COMPLEX NUMBERS

We introduce polar coordinates, in which we define the distance of a point (x, y) from the origin as r, which we call the absolute value of z, and the principal argument or *phase* of the complex number z is defined as

θ measured positive counterclockwise. Then

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{7}$$

and therefore

$$x + iy = r(\cos\theta + i\sin\theta), \tag{8}$$

where

$$r = \sqrt{x^2 + y^2}, \tag{9}$$
$$\theta = \tan^{-1}(y/x). \tag{10}$$

The modulus of the complex number $|z| = r > 0$ is defined uniquely for every value of x and y. However, the argument of a complex number θ can have any one of an infinite set of values and still satisfy (7). Thus θ, $\theta \pm 2\pi$, $\theta \pm 4\pi$, $\ldots$, $\theta \pm 2n\pi$, for $n = 1, 2, 3, \ldots$ can be any one of those values for which (7) is satisfied. As a matter of convention we often choose

$$0 \leq \theta \leq 2\pi, \tag{11}$$

and call it the *principal* value of argument z. Sometimes, the principal value of the argument z is defined as:

$$-\pi \leq \theta \leq -\pi. \tag{12}$$

5 ADDITION AND SUBTRACTION

Given two complex numbers z_1 and z_2, their sum is defined as

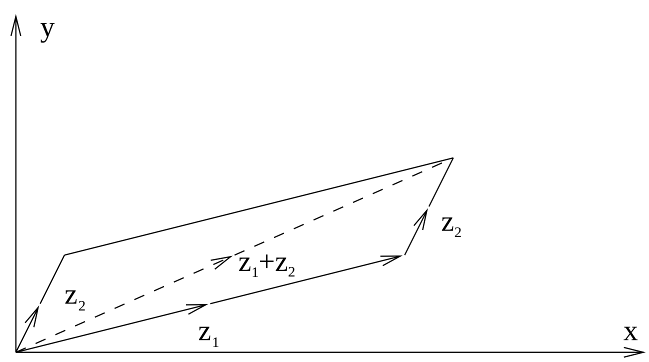

Figure 2.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2), \\ &= (x_1 + x_2) + i(y_1 + y_2), \\ &= (x_2 + x_1) + i(y_2 + y_1), \\ &= (x_2 + iy_2) + (x_1 + iy_1), \\ &= z_2 + z_1 \ , \end{aligned} \tag{13}$$

which is the "parallelogram law" of addition. Similarly,

$$\begin{aligned}(z_1+z_2)+z_3 &= [(x_1+iy_1)+(x_2+iy_2)]+(x_3+iy_3),\\ &= (x_1+x_2+x_3)+i(y_1+y_2+y_3),\\ &= (x_1+iy_1)+[(x_2+x_3)+i(y_2+y_3)],\\ &= (x_1+iy_1)+[(x_2+iy_2)+(x_3+iy_3)],\\ &= z_1+(z_2+z_3). \qquad (14)\end{aligned}$$

In general we can show that for complex numbers z_1, z_2 and z_3

$$(z_1+z_2)+z_3 = (z_2+z_3)+z_1 = (z_3+z_1)+z_2. \qquad (15)$$

Another property about the modulus of complex numbers is obvious from Fig.2. We note that z_1, z_2 and (z_1+z_2) form the three sides of a triangle. It is geometrically obvious from this figure that

$$|z_1+z_2| \le |z_1|+|z_2|. \qquad (16)$$

It is a statement of the fact that in general the modulus of one side of a triangle cannot be greater that the sum of the other two sides. Equality holds only when the sides of lengths $|z_1|$ and $|z_2|$ subtend an angle $\theta = 0$ between them. This is called the "triangle inequality" of complex numbers and we state it as follows:

$$|z_1+z_2| \le |z_1|+|z_2|. \qquad (17)$$

We can generalize this result to a quadrilateral with three sides z_1, z_2 and z_3. The fourth side is given by $z_1+z_2+z_3$ and the associated inequality is

$$|z_1+z_2+z_3| \le |z_1|+|z_2|+|z_3|. \qquad \diamond \qquad (18)$$

In general we have the triangle inequality

$$|z_1+z_2+\cdots+\cdots+z_n| \le |z_1|+|z_2|+\cdots+\cdots+|z_n|. \qquad \diamond \qquad (19)$$

Analytical proof of the triangle inequality is to be found in Section **9**.

6 MULTIPLICATION

Consider two complex numbers

$$\begin{aligned} z_1 &= x_1+iy_1, \qquad (20)\\ z_2 &= x_2+iy_2.\end{aligned}$$

Their product is defined as

$$\begin{aligned} z_1z_2 &= (x_1+iy_1)(x_2+iy_2),\\ &= (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1), \qquad (21)\end{aligned}$$

where we have defined $i\times i = (i)^2 = -1$.

This product representation of complex numbers take a useful form in polar coordinates. Thus, introducing polar coordinates, we find that if

$$z_1 = r_1(\cos\theta_1 + i\ \sin\theta_1), \qquad z_2 = r_2(\cos\theta_2 + i\ \sin\theta_2),$$

then

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2),$$
$$= r_1r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)\right]. \tag{22}$$

We now use the trigonometric identities [2]

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \tag{23}$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.$$

Substituting these identities in (22), we find

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{24}$$

Thus from here it follows that when multiplying two complex numbers with moduli r_1 and r_2 and arguments θ_1 and θ_2, we multiply the absolute values and add the arguments.

From here it also follows that

$$|z_1z_2| = r_1r_2|\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)|,$$
$$= r_1r_2|(\cos\phi + i\sin\phi)|, \tag{25}$$

where $\phi \equiv (\theta_1 + \theta_2)$. Now the absolute value of $(\cos\phi + i\sin\phi)$ is

$$|(\cos\phi + i\sin\phi)| = \sqrt{\cos^2\phi + \sin^2\phi} = 1, \tag{26}$$

and therefore

$$|z_1z_2| = r_1r_2. \tag{27}$$

We thus conclude that the *modulus of the product is equal to the product of the moduli.*

The argument of the product of two complex numbers is additive, because $\phi \equiv \theta_1 + \theta_2$, i.e.,

$$\arg\, z_1z_2 = \arg\, z_1 + \arg\, z_2. \qquad \diamond \tag{28}$$

Associative Law of Multiplication

We now establish the associative law of multiplication. Let z_1, z_2 and z_3 be three complex numbers. Then

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \tag{29}$$
$$z_3 = r_3[\cos\theta_3 + i\sin\theta_3]. \tag{30}$$

Upon multiplying the two we have

$$(z_1z_2)z_3 = r_1r_2r_3\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] \times (\cos\theta_3 + i\sin\theta_3),$$
$$= r_1r_2r_3[\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)], \tag{31}$$

[2] For a proof of these trigonometric identities see Appendix **A**

where we have used the trigonometric identities

$$\cos(\theta_1+\theta_2+\theta_3) = \cos(\theta_1+\theta_2)\cos\theta_3 - \sin(\theta_1+\theta_2)\sin\theta_3, \tag{32}$$

$$\sin(\theta_1+\theta_2+\theta_3) = \sin(\theta_1+\theta_2)\cos\theta_3 + \cos(\theta_1+\theta_2)\sin\theta_3. \tag{33}$$

It is easy to see that

$$\begin{aligned} z_2z_3 &= r_2r_3[\cos(\theta_2+\theta_3) + i\sin(\theta_2+\theta_3)], \\ z_1 &= r_1(\cos\theta_1 + i\sin\theta_1), \end{aligned} \tag{34}$$

and therefore upon multiplying the two we obtain

$$\begin{aligned} (z_2z_3)z_1 &= r_1r_2r_3\left[\cos(\theta_2+\theta_3) + i\sin(\theta_2+\theta_3)\right](\cos\theta_1 + i\sin\theta_1), \\ &= r_1r_2r_3\left[\cos(\theta_2+\theta_3+\theta_1) + i\sin(\theta_2+\theta_3+\theta_1)\right], \end{aligned} \tag{35}$$

where

$$\cos(\theta_2+\theta_3+\theta_1) = \cos(\theta_2+\theta_3)\cos\theta_1 - \sin(\theta_2+\theta_3)\sin\theta_1, \tag{36}$$

$$\sin(\theta_2+\theta_3+\theta_1) = \sin(\theta_2+\theta_3)\cos\theta_1 + \cos(\theta_2+\theta_3)\sin\theta_1. \tag{37}$$

Comparing (31) and (35), we find the associative law of multiplication

$$(z_1z_2)z_3 = z_1(z_2z_3). \qquad \diamond \tag{38}$$

Distributive law of multiplication

To establish the distributive law of multiplication, we assume three distinct complex numbers z_1, z_2 and z_3. Using polar representation we find

$$\begin{aligned} z_2 + z_3 &= r_2(\cos\theta_2 + i\sin\theta_2) + r_3(\cos\theta_3 + i\sin\theta_3), \\ z_1 &= r_1(\cos\theta_1 + i\sin\theta_1). \end{aligned} \tag{39}$$

Therefore, the product is

$$\begin{aligned} z_1(z_2+z_3) &= r_1(\cos\theta_1 + i\sin\theta_1)[r_2(\cos\theta_2 + i\sin\theta_2) + r_3(\cos\theta_3 + i\sin\theta_3)], \\ &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) + r_1r_3(\cos\theta_1 + i\sin\theta_1)(\cos\theta_3 + i\sin\theta_3). \end{aligned} \tag{40}$$

Now for any fixed set of integers m and n

$$\begin{aligned} &(\cos\theta_m + i\sin\theta_m)(\cos\theta_n + i\sin\theta_n) \\ &\qquad = (\cos\theta_m\cos\theta_n - \sin\theta_m\sin\theta_n) + i(\cos\theta_m\sin\theta_n + \sin\theta_m\cos\theta_n), \\ &\qquad = \cos(\theta_m+\theta_n) + i\sin(\theta_m+\theta_n), \qquad \text{for } m \neq n. \end{aligned} \tag{41}$$

Substituting this in (40) we find for $m = 1$, $n = 2$, and $m = 1$, $n = 3$

$$\begin{aligned} z_1(z_2+z_3) &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) + r_1r_3(\cos\theta_1 + i\sin\theta_1)(\cos\theta_3 + i\sin\theta_3), \\ &= z_1z_2 + z_1z_3. \qquad \diamond \end{aligned} \tag{42}$$

7 CONJUGATE NUMBERS

Given a complex number

$$z = x + iy, \tag{43}$$

its complex conjugate is defined as the complex number

$$\overline{z} = x - iy. \tag{44}$$

Thus the number $\overline{z}$, represented by the point $(x, -y)$, is the reflection of the point (x, y) about the real axis.

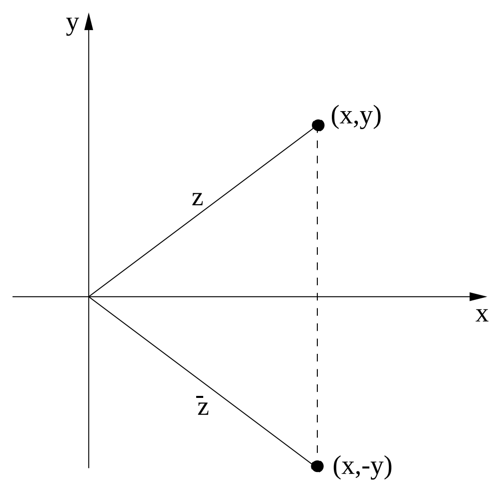

Figure 3.

The absolute value of the conjugate is

$$|\overline{z}| = |x - iy| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|, \tag{45}$$

and this is true for all points z in the complex plane.

1. If we have two complex numbers z_1 and z_2, then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)}, \\ &= \overline{(x_1 + x_2) + i(y_1 + y_2)}, \\ &= (x_1 + x_2) - i(y_1 + y_2), \\ &= (x_1 - iy_1) + (x_2 - iy_2), \\ &= \overline{z_1} + \overline{z_2}. \qquad \diamond \end{aligned} \tag{46}$$

   Hence, *the conjugate of the sum is equal to the sum of the conjugates.*

2. In a similar manner we can show that

$$\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}, \qquad \diamond \tag{47}$$

   i.e., *the conjugate of the difference is equal to the difference of the conjugates.*

3. We now consider conjugate of the product of two complex numbers. Thus

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2), \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned} \tag{48}$$

The conjugate of the product is

$$\begin{aligned} \overline{z_1 z_2} &= r_1 r_2[\cos(\theta_1+\theta_2) - i\sin(\theta_1+\theta_2)], \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) - i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)], \\ &= r_1 r_2[\cos\theta_1(\cos\theta_2 - i\sin\theta_2) - i\sin\theta_1(\cos\theta_2 - i\sin\theta_2)], \\ &= r_1 r_2(\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2), \\ &= [r_1(\cos\theta_1 - i\sin\theta_1)][r_2(\cos\theta_2 - i\sin\theta_2)], \\ &= \overline{z_1}\ \overline{z_2}. \qquad \diamond \end{aligned} \tag{49}$$

Thus the *conjugate of the product is equal to the product of the conjugates.*

4. Consider now the conjugate of the quotient of two complex numbers. Let

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right). \tag{50}$$

Multiplying both the numerator and denominator on the right hand side of (49) by $(\cos\theta_2 - i\sin\theta_2)$, we get

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2), \\ &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]. \end{aligned} \tag{51}$$

Now taking the conjugate of both sides of the above equation, we find

$$\begin{aligned} \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) - i\sin(\theta_1-\theta_2)], \\ &= \frac{r_1}{r_2}(\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2), \\ &= \frac{r_1}{r_2}\frac{(\cos\theta_1 - i\sin\theta_1)}{(\cos\theta_2 - i\sin\theta_2)}, \\ &= \frac{\overline{z_1}}{\overline{z_2}}. \qquad \diamond \end{aligned} \tag{52}$$

Thus the *conjugate of a complex quotient function is equal to the quotient of complex conjugate functions.*

5. Sum of a complex function and its conjugate

Let $z = x + iy$ and $\overline{z} = x - iy$. Then adding the two we get

$$2x = z + \overline{z}\ , \tag{53}$$

and subtracting the two we find

$$2iy = z - \overline{z}\ . \tag{54}$$

Hence, we have the two identities

$$\mathcal{R}e[z] \equiv \frac{1}{2}(z + \overline{z}), \tag{55}$$

$$\mathcal{I}m[z] \equiv \frac{1}{2i}(z - \overline{z}), \tag{56}$$

where

$$\mathcal{R}e[z] \text{ :- Real part of } z, \tag{57}$$

$$\mathcal{I}m[z] \text{ :- Imaginary part of } z. \tag{58}$$

Between the modulus and conjugate of a complex number there exists an important identity. Thus corresponding to the complex number z and its conjugate $\overline{z}$, their product is

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy), \\ &= (x^2 + y^2), \\ &= |z^2|. \end{aligned} \tag{59}$$

Thus if we multiply a complex number by its conjugate, we get the real number $|z^2|$. This formula is also useful in representing the quotient of two complex functions in terms of a single complex function.

Example 1. Let $z = 1/(3+i)$. On multiplying the numerator and denominator by the conjugate $(3-i)$, we get

$$z = \frac{1}{(3+i)}\frac{(3-i)}{(3-i)} = \frac{1}{10}(3-i). \qquad \diamond \tag{60}$$

Example 2. Simplify the complex number $\frac{1}{(3+4i)}\frac{1}{(2+i)}$.

This can be written as

$$\begin{aligned} \frac{1}{(3+4i)(2+i)} &= \frac{1}{(2+11i)}, \\ &= \frac{1}{(2+11i)}\frac{(2-11i)}{(2-11i)}, \\ &= \frac{2-11i}{2^2+11^2}, \\ &= \frac{1}{125}(2-11i). \qquad \diamond \end{aligned} \tag{61}$$

8 OTHER PROPERTIES OF MODULI

Using the fundamental property

$$z\overline{z} = |z|^2, \tag{62}$$

we can easily establish several other properties:

1.

$$|z_1 z_2| = |z_1||z_2|. \tag{63}$$

To derive this property we write

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)\overline{(z_1 z_2)}\ , \\ &= (z_1 z_2)(\overline{z_1}\ \overline{z_2}), \\ &= (z_1 \overline{z_1})(z_2 \overline{z_2}), \\ &= |z_1|^2 |z_2|^2, \\ &= (|z_1||z_2|)^2. \end{aligned}$$

Therefore

$$|z_1 z_2| = |z_1||z_2|, \qquad \diamond$$

since modulus is always positive.

2.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{64}$$

To derive this property we write

$$\begin{aligned} \left|\frac{z_1}{z_2}\right|^2 &= \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)}, \\ &= \left(\frac{z_1}{z_2}\right)\left(\frac{\overline{z_1}}{\overline{z_2}}\right), \\ &= \frac{z_1 \overline{z_1}}{z_2 \overline{z_2}}\ , \\ &= \frac{|z_1|^2}{|z_2|^2}\ . \end{aligned}$$

Since modulus is never negative, we immediately deduce the *quotient theorem*

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \qquad \diamond \tag{65}$$

Example 1. Given two complex numbers

$$\begin{aligned} z_1 &= 2 + 3i, \\ z_2 &= 3 + 4i, \end{aligned}$$

evaluate $|z_1 z_2|$ and $|z_1/z_2|$ and show that these are equal to $|z_1||z_2|$ and $|z_1|/|z_2|$, respectively.

$$\begin{aligned} z_1 z_2 &= (2 + 3i)(3 + 4i), \\ &= (6 - 12) + i(2 \times 4 + 3 \times 3), \\ &= -6 + 17i. \end{aligned}$$

Therefore

$$\begin{aligned}|z_1z_2|^2 &= (-6+17i)(-6-17i),\\ &= (-6)^2+(17)^2,\\ &= 325 = 25\times 13.\end{aligned}$$

Hence

$$|z_1z_2| = 5\sqrt{13}.$$

Also,

$$|z_1| = |2+3i| = \sqrt{4+9} = \sqrt{13}\ ,$$

$$|z_2| = |3+4i| = \sqrt{9+16} = 5,$$

and therefore

$$|z_1||z_2| = 5\sqrt{13}\ .$$

Hence,

$$|z_1z_2| = |z_1||z_2|. \qquad \diamond$$

Similarly

$$\frac{z_1}{z_2} = \frac{2+3i}{3+4i}.$$

To find the modulus of the quotient $\left|z_1/z_2\right|$ we write

$$\begin{aligned}\left|\frac{z_1}{z_2}\right|^2 &= \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)},\\ &= \left(\frac{z_1}{z_2}\right)\left(\frac{\overline{z_1}}{\overline{z_2}}\right),\\ &= \frac{z_1\overline{z_1}}{z_2\overline{z_2}}\ ,\\ &= \frac{(2+3i)}{(3+4i)}\frac{(2-3i)}{(3-4i)},\\ &= \frac{2^2+3^2}{3^2+4^2} = \frac{13}{25}.\end{aligned}$$

Since modulus is never negative, we find

$$\left|\frac{z_1}{z_2}\right| = \frac{\sqrt{13}}{5} = \frac{|z_1|}{|z_2|}. \qquad \diamond$$

Example 2. Show that

$$|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|.$$

Using the product rule of moduli we find

$$\begin{aligned} |(2\overline{z}+5)(\sqrt{2}-i)| &= |2\overline{z}+5||\sqrt{2}-i|, \\ &= |2\overline{z}+5|\sqrt{(\sqrt{2})^2+1}\ , \\ &= \sqrt{3}|2\overline{z}+5|. \end{aligned}$$

Now $|z| = |\overline{z}|$ and in general for any complex number $az+b$, $|az+b| = |a\overline{z}+b|$. This follows from the fact that

$$\begin{aligned} |az+b| &= |(ax+b)+iay| = \sqrt{(ax+b)^2+(ay)^2}, \\ |a\overline{z}+b| &= |(ax+b)-iay| = \sqrt{(ax+b)^2+(ay)^2}. \end{aligned}$$

Hence $|2\overline{z}+5| = |2z+5|$ and therefore

$$\begin{aligned} |(2\overline{z}+5)(\sqrt{2}-i)| &= \sqrt{3}|2\overline{z}+5|, \\ &= \sqrt{3}|2z+5|. \qquad \text{Q.E.F.} \end{aligned}$$

We can verify this by direct computation. To see this we write

$$\begin{aligned} (2\overline{z}+5)(\sqrt{2}-i) &= [(2x+5)-2iy](\sqrt{2}-i), \\ &= [\sqrt{2}(2x+5)-2y] - i[(2x+5)+2\sqrt{2}y]. \end{aligned}$$

Hence

$$\begin{aligned} |(2\overline{z}+5)(\sqrt{2}-i)|^2 &= [\sqrt{2}(2x+5)-2y]^2 + [(2x+5)+2\sqrt{2}y]^2, \\ &= 3[(2x+5)^2+4y^2], \\ &= 3[(2x+5)+2iy][(2x+5)-2iy], \\ &= 3[2(x+iy)+5][2(x-iy)+5], \\ &= 3[2z+5][2\overline{z}+5], \\ &= 3(2z+5)(\overline{2z+5}), \\ &= 3|2z+5|^2, \end{aligned}$$

or

$$|(2\overline{z}+5)(\sqrt{2}-i)| = \sqrt{3}|2z+5|. \qquad \diamond$$

9 TRIANGLE INEQUALITY

Given two complex numbers z_1 and z_2, triangle inequality states that

$$|z_1+z_2| \leq |z_1| + |z_2|.$$

This inequality is obvious geometrically, as was previously stated in Section 5. We now give an analytical proof. This triangle inequality states that the length of the hypotenuse of a triangle is either less than or equal to the sum of the lengths of the remaining two sides. Equality holds only when both z_1 and z_2 lie on the same line.

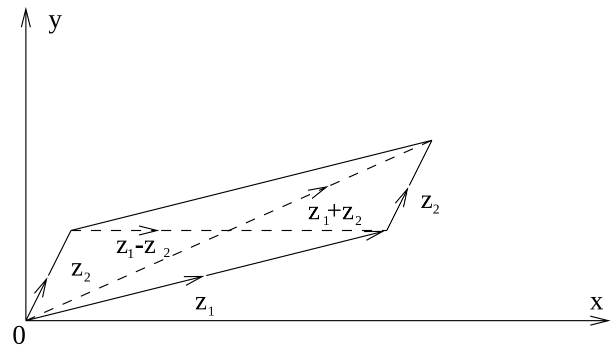

Figure 4.

PROOF:

$$\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)(\overline{z_1+z_2}),\\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}),\\
&= z_1\overline{z_1}+z_2\overline{z_2}+(z_1\overline{z_2}+\overline{z_1}z_2),\\
&= |z_1|^2+|z_2|^2+(z_1\overline{z_2}+(\overline{z_1\overline{z_2}}),\\
&= |z_1|^2+|z_2|^2+2\mathcal{R}e[z_1\overline{z_2}],\\
&\leq |z_1|^2+|z_2|^2+2|z_1\overline{z_2}|,\\
&= |z_1|^2+|z_2|^2+2|z_1||\overline{z_2}|,\\
&= |z_1|^2+|z_2|^2+2|z_1||z_2|,\\
&= (|z_1|+|z_2|)^2.
\end{aligned}$$

Hence,

$$|z_1+z_2| \leq |z_1|+|z_2|, \tag{66}$$

since modulus of a complex number is always positive. ■

Triangle inequality can now be generalized to a sum of a finite numbers of complex numbers. Consider three complex numbers z_1, z_2 and z_3. Then using triangle inequality repeatedly, we can write

$$\begin{aligned}
|z_1+z_2+z_3| &\leq |z_1+z_2|+|z_3|,\\
&\leq |z_1|+|z_2|+|z_3|. \qquad (67)
\end{aligned}$$

Using mathematical induction, the generalization is

$$|z_1+z_2+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|, \quad (n=2,3,\cdots). \qquad \diamond \tag{68}$$

With the help of triangle identity, we now prove another useful identity:

$$||z_1|-|z_2|| \leq |z_1+z_2|. \tag{69}$$

PROOF : To prove this identity we note that

$$\begin{aligned}
|z_1| &= |(z_1-z_2)+z_2|,\\
&\leq |z_1-z_2|+|z_2|, \qquad (70)
\end{aligned}$$

where in the last step we have made use of the triangle inequality.Hence,

$$|z_1| - |z_2| \le |z_1 - z_2|. \qquad \diamond \tag{71}$$

We can similarly show that

$$\begin{aligned} |z_2| &= |(z_2 - z_1) + z_1|, \\ &\le |z_2 - z_1| + |z_1|, \end{aligned} \tag{72}$$

and therefore

$$\begin{aligned} |z_2| - |z_1| &\le |z_2 - z_1|, \\ &= |z_1 - z_2|, \end{aligned} \tag{73}$$

because $|z_1 - z_2| = |z_2 - z_1|$. Hence, we have two important results

$$|z_1| - |z_2| \le |z_1 - z_2|, \tag{74}$$

and

$$|z_2| - |z_1| \le |z_1 - z_2|. \tag{75}$$

Obviously the first statement is true for $|z_1| \ge |z_2|$, and the second statement is valid for $|z_2| \ge |z_1|$. Combining the two, we have the general result

$$||z_1| - |z_2|| \le |z_1 - z_2|. \qquad \blacksquare \tag{76}$$

Some examples using conjugate and modulus of complex numbers:

Example 1. Evaluate $\overline{(2+i)^2}$.

$$\begin{aligned} (2+i)^2 &= (2+i)(2+i), \\ &= (4-1) + (2i + 2i), \\ &= 3 + 4i. \end{aligned}$$

Therefore the conjugate is $(3 - 4i)$. $\diamond$

Example 2. Evaluate $\overline{(\overline{z} + 4i)}$.

$$\begin{aligned} (\overline{z} + 4i) &= (x - iy) + 4i, \\ &= x + i(4 - y). \end{aligned}$$

Hence, the conjugate is

$$\begin{aligned} \overline{(\overline{z} + 4i)} &= x - i(4 - y), \\ &= (x + iy) - 4i, \\ &= z - 4i. \qquad \diamond \end{aligned}$$

Note that we can make use of the fundamental property that conjugate of the conjugate of a complex number is the complex number itself; i.e., $\overline{\overline{z}} \equiv z$. Thus, we can write

$$\begin{aligned} \overline{(\overline{z} + 4i)} &= \overline{\overline{z}} - 4i, \\ &= z - 4i, \end{aligned}$$

since $\bar{\bar{z}} \equiv z$.

Example 3. Show that if $|z_3| \neq |z_4|$, then

$$\frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}\,.$$

We are dealing here with the absolute value of a complex number in quotient form, and we are interested in finding the maximum value of the modulus of the quotient function. This maximum will be achieved when the modulus of the numerator reaches a maximum and modulus of the denominator reaches a minimum.

Using triangle inequality we find that

$$\begin{aligned} \max|z_1+z_2| &\leq |z_1|+|z_2|, \quad \text{and} \\ \min|z_3+z_4| &\leq ||z_3|-|z_4||. \end{aligned}$$

Therefore,

$$\max\frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

As an example let

$$\begin{aligned} z_1 &= 3+4i, \;\; z_3 = 6+i, \\ z_2 &= 1+3i, \;\; z_4 = 3+2i. \end{aligned} \tag{77}$$

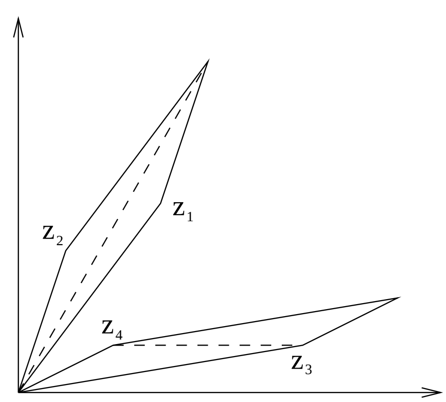

Figure 5.

$$\begin{aligned} |z_1+z_2| &= |(3+4i)+(1+3i)|, \\ &\leq |3+4i|+|1+3i|, \\ &= 5+\sqrt{10}. \end{aligned}$$

$$\begin{aligned} |z_3+z_4| &= |(6+i)+(3+2i)|, \\ &\leq |6+i|-|3+2i|, \\ &= \sqrt{37}-\sqrt{13}. \end{aligned}$$

Therefore,

$$\max\frac{|z_1+z_2|}{|z_3+z_4|} \le \frac{5+\sqrt{10}}{\sqrt{37}-\sqrt{13}}, \\ = \frac{1}{24}(5+\sqrt{10})(\sqrt{37}+\sqrt{13})\,. \qquad \diamond$$

Example 4. Let $z_1 = (2-i)$ and $z_2 = (\sqrt{3}-2i)$.

1. Evaluate $|z_1^2 + 2z_2|$.

$$\begin{aligned} z_1^2 &= (2-i)^2, \\ &= 3-4i. \\ 2z_2 &= 2(\sqrt{3}-2i). \\ &= 2\sqrt{3}-4i. \end{aligned}$$

Therefore adding the two

$$\begin{aligned} z_1^2 + 2z_2 &= (3+2\sqrt{3}) - 8i\,, \\ |z_1^2+2z_2| &= \sqrt{[(3+2\sqrt{3})^2+8^2]}\,, \\ &= \sqrt{85+12\sqrt{3}}\,. \qquad \diamond \end{aligned}$$

2. Evaluate $|z_1\overline{z_2} + z_2\overline{z_1}|$.

We first note that $\overline{z_1\overline{z_2}} = \overline{z_1}z_2$. Therefore, $z_1\overline{z_2}+z_2\overline{z_1} = z_1\overline{z_2}+\overline{z_1\overline{z_2}} = 2\mathcal{R}e[z_1\overline{z_2}+z_2\overline{z_1}]$, and consequently

$$|z_1\overline{z_2}+z_2\overline{z_1}| = 2\mathcal{R}e[z_1\overline{z_2}+z_2\overline{z_1}].$$

Now

$$\begin{aligned} z_1\overline{z_2} &= (2-i)(\sqrt{3}+2i), \\ &= (2\sqrt{3}+2) + i(4-\sqrt{3}), \end{aligned}$$

and

$$\overline{z_1\overline{z_2}} = (2\sqrt{3}+2) - i(4-\sqrt{3}).$$

Therefore

$$\begin{aligned} |z_1\overline{z_2}+z_2\overline{z_1}| &= 2(2\sqrt{3}+2), \\ &= 4(\sqrt{3}+1). \qquad \diamond \end{aligned}$$

3. Evaluate $\frac{1}{2}(z/\overline{z}+\overline{z}/z)$ where $z = -2+3i$.

We first note that

$$\overline{\left(\frac{z}{\overline{z}}\right)} = \frac{\overline{z}}{\overline{\overline{z}}} = \frac{\overline{z}}{z}.$$

Therefore

$$\frac{1}{2}\left(\frac{z}{\overline{z}}+\frac{\overline{z}}{z}\right) = \frac{1}{2}\left[\left(\frac{z}{\overline{z}}\right)+\overline{\left(\frac{z}{\overline{z}}\right)}\right], \\ = \mathcal{R}e\left(\frac{z}{\overline{z}}\right).$$

Now

$$\begin{aligned} \frac{z}{\overline{z}} &= \frac{-2+3i}{-2-3i}, \\ &= \frac{(2-3i)(2-3i)}{2^2+3^2}, \\ &= \frac{1}{13}[(4-9)-12i], \\ &= -\frac{1}{13}(5+12i). \end{aligned}$$

Therefore

$$\frac{1}{2}\left(\frac{z}{\overline{z}}+\frac{\overline{z}}{z}\right) = \mathcal{R}e\left(\frac{z}{\overline{z}}\right) = -\frac{5}{13}. \qquad \diamond$$

4. Evaluate $\overline{(z_1+z_3)(z_2-z_3)}$ where $(z_1, z_2, z_3) = (2-i\ ,\sqrt{3}-2i\ ,-2+3i)$.
We first note that

$$\begin{aligned} \overline{(z_1+z_3)(z_2-z_3)} &= (\overline{z_1+z_3})(\overline{z_2-z_3}), \\ &= (\overline{z_1}+\overline{z_3})(\overline{z_2}-\overline{z_3}). \end{aligned}$$

$$\begin{aligned} \overline{z_1}+\overline{z_3} &= (2+i)+(-2-3i), \\ &= -2i\ ; \\ \overline{z_2}-\overline{z_3} &= (\sqrt{3}+2i)-(-2-3i), \\ &= (2+\sqrt{3})+5i. \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{(z_1+z_3)(z_2-z_3)} &= -i2[(2+\sqrt{3})+5i], \\ &= 10-i2(2+\sqrt{3}). \end{aligned}$$

To verify this result we note that

$$\begin{aligned} z_1+z_3 &= (2-i)+(-2+3i), \\ &= 2i\ ; \\ z_2-z_3 &= (\sqrt{3}-2i)-(-2+3i), \\ &= (2+\sqrt{3})-5i. \end{aligned}$$

Therefore, the product is

$$\begin{aligned} (z_1+z_3)(z_1-z_3) &= i2[(2+\sqrt{3})-5i], \\ &= 10+i2(2+\sqrt{3}), \end{aligned}$$

and its conjugate is $10-i2(2+\sqrt{3})$. Q.E.F.[3]

[3] Latin: Quod Erat Faciendum *(fā–kE–en–dum)* = *which was to be done.*

Example 5. Evaluate $z = (\frac{1+i}{1-i})^2 - 3(\frac{1-i}{1+i})^3$
This can be written as

$$\begin{aligned}
z &= \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^2 - 3\left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^3, \\
&= \frac{(1+i)^4}{2^2} - 3\frac{(1-i)^6}{2^3}, \\
&= \frac{(1-1+2i)^2}{4} - \frac{3}{8}(1-1-2i)^3, \\
&= (i)^2 + \frac{3}{8} \times 8(i)^3, \\
&= -(1+3i). \qquad \text{Q.E.F.}
\end{aligned}$$

Example 6. Simplify the equation of the curve $|\frac{z-3}{z+3}| = 2$.

Taking the conjugate we find $|\frac{\overline{z}-3}{\overline{z}+3}| = 2$. Now taking the product of the two equations we get on simplifying

$$|z-3||\overline{z}-3| = 4|z+3||\overline{z}+3|.$$

or

$$|z\overline{z} - 3z - 3\overline{z} + 9| = 4|z\overline{z} + 3z + 3\overline{z} + 9|.$$

The quantities inside the absolute value signs are real because $z\overline{z}$ and $(z+\overline{z})$ are both real. Therefore

$$z\overline{z} - 3(z+\overline{z}) + 9 = 4z\overline{z} + 12(z+\overline{z}) + 36,$$

or

$$3[z\overline{z} + 5(z+\overline{z}) + 9] = 0.$$

This can be written as

$$(z+5)(\overline{z}+5) - 25 + 9 = 0,$$

or

$$(z+5)(\overline{z}+5) = |z+5|^2 = 4^2. \qquad \diamond$$

Hence we obtain $|z+5| = 4$, which is the equation of a circle of radius $r = 4$ with center at $z = -5$, i.e.,the center of the circle is located at the point $(x,\ y) = (-5,\ 0)$.

10 POLAR FORM OF COMPLEX NUMBERS

We can represent a complex number by a point (x, y) in the complex plane (also called *the Argand plane*). Conversely, to every point in the Argand plane there corresponds a complex number. For this reason the complex number z is also called the point z.

Given a point $z = (x+iy)$, the distance of this point from the origin is defined as

$$|z| = \sqrt{|z||\overline{z}|} = \sqrt{x^2+y^2},$$

where we take only the positive value before the radical sign, because distance of a point from the origin is a positive quantity.

Now consider two distinct points z_1 and z_2 in the Argand plane. Let the representations of these two points be:

$$z_1 = x_1 + iy_1 \ , \qquad z_2 = x_2 + iy_2.$$

Then the distance between these two points is

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

In the Argand plane, consider a typical point $P : (x, y)$, which represents a complex number $z = (x + iy)$.

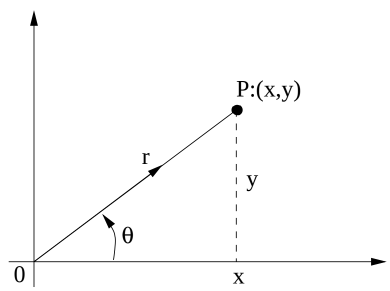

Figure 6.

For the point P, the cartesian coordinates are (x, y) and the polar coordinates are (r, θ). These two coordinates are related to each other by a single-valued transformation

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \qquad r > 0 \ , \ \ 0 \leq \theta < 2\pi. \end{aligned}$$

Because the *Jacobian* $r > 0$, the inverse of the transformation exists and is given by

$$\begin{aligned} r &= |z| = \sqrt{|z\overline{z}|} = \sqrt{x^2 + y^2}, \\ \theta &= \tan^{-1}(y/x). \end{aligned}$$

Here $r = |z|$ is called the modulus or the absolute value of the complex number; and θ is called the argument or phase of the complex number. Modulus of a complex number z is usually written as $|z|$; and the argument θ or the phase of a complex number z is written as $\arg z$. The angle θ which the line OP makes with the positive x-axis is measured positive counterclockwise and its principal range is $0 \leq \theta < 2\pi$.

We can write the complex number in the polar form:

$$\begin{aligned} z &= x + iy, \\ &= r(\cos\theta + i\sin\theta), \qquad 0 < r < \infty, \ \ 0 \leq \theta < 2\pi. \end{aligned}$$

It is often found convenient to write

$$\begin{aligned} \cos\theta + i\sin\theta &\equiv \mathrm{cis}\theta, \\ \cos\theta - i\sin\theta &\equiv \overline{\mathrm{cis}}\theta, \end{aligned}$$

where $\mathrm{cis}\theta \ \overline{\mathrm{cis}}\theta = 1$.

In polar coordinates the principal value of the range of θ is $[0, 2\pi)$. However, one can use any other range of θ of length 2π. Thus $-\pi \leq \theta < \pi$ is also a suitable principal range of θ.

11 de MOIVRE'S THEOREM

The algebra of complex numbers simplifies considerably if we make use of polar representation of complex numbers and Euler's exponential representation of cisθ.
Consider two complex numbers in polar form:

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1), \qquad & r_i > 0, \quad 0 \le \theta_i < 2\pi; \\ z_2 &= r_2(\cos\theta_2 + i\sin\theta_2), \qquad & i = 1, 2. \end{aligned}$$

The product of these two complex numbers is

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2), \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)], \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \end{aligned}$$

where $r_1 r_2 > 0$, $0 \le (\theta_1 + \theta_2) < 2\pi$. It is obvious that in the last step we have used the trigonometric identity

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \end{aligned}$$

The product of the conjugate numbers $\overline{z_1}$ and $\overline{z_2}$ is

$$\begin{aligned} \overline{z_1 z_2} &= r_1 r_2(\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2), \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) - i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)], \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) - i\sin(\theta_1 + \theta_2)], \end{aligned}$$

which is indeed the conjugate of $z_1 z_2$. Here again $r_1 r_2 > 0$ and $0 \le (\theta_1 + \theta_2) < 2\pi$.

The quotient of two complex numbers z_1 and z_2 in polar form is:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}, \\ &= \frac{r_1}{r_2}\,(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2), \\ &= \frac{r_1}{r_2}\,[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)], \\ &= \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \end{aligned}$$

where $0 < r_1/r_2 < \infty$, $0 \le (\theta_1 - \theta_2) < 2\pi$.

The quotient of the complex conjugate numbers $\overline{z_1}$ and $\overline{z_2}$ is defined as

$$\begin{aligned} \frac{\overline{z_1}}{\overline{z_2}} &= \frac{r_1}{r_2}\,\frac{(\cos\theta_1 - i\sin\theta_1)}{(\cos\theta_2 - i\sin\theta_2)}, \\ &= \frac{r_1}{r_2}\,(\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2), \\ &= \frac{r_1}{r_2}\,[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)], \\ &= \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)], \end{aligned}$$

where $0 < r_1/r_2 < \infty$, $0 \leq (\theta_1 - \theta_2) < 2\pi$.

We also note from these results that

$$\frac{\overline{z_1}}{\overline{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)}.$$

Thus the product and quotient of complex numbers are defined as

$$\begin{aligned}
z_1 z_2 &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)], \\
\overline{z_1 z_2} &= r_1 r_2[\cos(\theta_1 + \theta_2) - i\sin(\theta_1 + \theta_2)], \\
\frac{z_1}{z_2} &= (\frac{r_1}{r_2})[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \\
\frac{\overline{z_1}}{\overline{z_2}} &= (\frac{r_1}{r_2})[\cos(\theta_1 - \theta_2) - i\sin(\theta_1 - \theta_2)].
\end{aligned}$$

We can now generalize these formulas for the product and quotient of complex numbers. By using mathematical induction one can show that

$$\begin{aligned}
z_1 z_2 \dots z_n &= r_1 r_2 \dots r_n[\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)], \\
\overline{z_1 z_2} \dots \overline{z_n} &= r_1 r_2 \dots r_n[\cos(\theta_1 + \theta_2 + \dots + \theta_n) - i\sin(\theta_1 + \theta_2 + \dots + \theta_n)]. \qquad \diamond
\end{aligned}$$

To prove by induction, we first assume that for some arbitrary choice of positive integer n, the product

$$\prod_{k=1}^{n} z_k = r_1 r_2 \dots r_n[\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)],$$

is true. Here the symbol $\prod_{k=1}^{n}$ is the product symbol and $\prod_{k=1}^{n} z_k \equiv z_1 z_2 \dots z_n$. Then multiplying both sides of the identity by z_{n+1}, we get

$$\begin{aligned}
\prod_{k=1}^{n+1} z_k &= r_1 r_2 \dots r_n r_{n+1}[\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n)] \\
&\times (\cos\theta_{n+1} + i\sin\theta_{n+1}), \\
&= r_1 r_2 \dots r_n r_{n+1}\{[\cos(\sum_{k=1}^{n} \theta_k)\cos\theta_{n+1} - \sin(\sum_{k=1}^{n} \theta_k)\sin\theta_{n+1}] + \\
&i\{[\sin(\sum_{k=1}^{n} \theta_k)\cos\theta_{n+1} + \cos(\sum_{k=1}^{n} \theta_k)\sin\theta_{n+1}]\}, \\
&= r_1 r_2 \dots r_n r_{n+1}\cos[(\theta_1 + \theta_2 + \dots + \theta_n) + \theta_{n+1}] + i\sin[(\theta_1 + \theta_2 + \dots + \theta_n) + \theta_{n+1}].
\end{aligned}$$

Now if the product rule is true for some arbitrary positive integer n, then as shown above it is also true for the real positive integer $(n + 1)$. This formula is certainly true for $n = 0$ for which

$$\prod_{k=1}^{1} z_k = r_1(\cos\theta_1 + i\sin\theta_1),$$

and is also true for $n = 1$ for which

$$\prod_{k=1}^{2} z_k = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Therefore it follows that the product rule is true for all real positive integers n. A generalization of the product rule for complex conjugate numbers can be sketched using similar arguments.

For the product of the quotients of complex numbers we can similarly show that

$$\begin{aligned}\frac{z_1}{z_2}\frac{z_3}{z_4}\cdots\frac{z_{n-1}}{z_n} = \frac{r_1}{r_2}\frac{r_3}{r_4}\cdots\frac{r_{n-1}}{r_n}&[\cos(\theta_1-\theta_2)+\cdots+\cos(\theta_{n-1}-\theta_n)]\\ &+i[\sin(\theta_1-\theta_2)+\cdots+\sin(\theta_{n-1}-\theta_n)]. \qquad \diamond\end{aligned}$$

We are now in a position to prove DeMoivre's Theorem for complex numbers. When $z_1 = z_2 = z_3 = \ldots = z_n$, we find

$$\begin{aligned} z^2 &= [r(\cos\theta + i\sin\theta)]^2 = r^2[\cos 2\theta + i\sin 2\theta],\\ z^3 &= [r(\cos\theta + i\sin\theta)]^3 = r^3[\cos 3\theta + i\sin 3\theta],\\ &\vdots\\ z^n &= [r(\cos\theta + i\sin\theta)]^n = r^n[\cos n\theta + i\sin n\theta]. \qquad \diamond\end{aligned}$$

This is often called **de Moivre's Theorem**[4] for complex numbers.

We can similarly show that if

$$\overline{z} = r(\cos\theta - i\sin\theta),$$

then

$$(\overline{z})^n = [r(\cos\theta - i\sin\theta)]^n = r^n[\cos n\theta - i\sin n\theta]. \qquad \diamond$$

Similarly

$$\begin{aligned}\left(\frac{1}{z}\right)^n &= \left[\frac{\cos\theta - i\sin\theta}{r}\right]^n = \frac{(\cos n\theta - i\sin n\theta)}{r^n}, \qquad \diamond\\ \left(\frac{1}{\overline{z}}\right)^n &= \left[\frac{\cos\theta + i\sin\theta}{r}\right]^n = \frac{(\cos n\theta + i\sin n\theta)}{r^n}. \qquad \diamond\end{aligned}$$

12 THE ROOTS OF A COMPLEX NUMBER

Consider a complex number

$$\omega \equiv (a + ib).$$

If

$$\omega^n = z,$$

then ω is said to be the n-th root of z, where z is the complex number $z = (x + iy)$.

[4] de Moivre, Abraham (1667–1754).

Using polar representation of complex numbers we can write

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta), \\
&= r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)], \qquad k = 0, 1, 2, \ldots
\end{aligned}
$$

where we have made use of the fundamental property that trigonometric functions are 2π-periodic, i.e.,

$$
\begin{aligned}
\cos(\theta + 2k\pi) &= (-1)^{2k}\cos\theta = \cos\theta, \quad k = 0, 1, 2, \ldots \\
\sin(\theta + 2k\pi) &= (-1)^{2k}\sin\theta = \sin\theta, \quad k = 0, 1, 2, \cdots.
\end{aligned}
$$

Using DeMoivre's Theorem, the n roots of the complex number z are given by

$$
\begin{aligned}
z^{\frac{1}{n}} &= r^{\frac{1}{n}}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{\frac{1}{n}}, \\
&= r^{\frac{1}{n}}[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})]^{\frac{1}{n}}, \quad k = 0, 1, \ldots, (n-1).
\end{aligned}
$$

Thus the n roots of a complex number for $k = 0, 1, 2, \ldots, (n-1)$ are:

$$
\begin{aligned}
z_{(0)}^{\frac{1}{n}} &= r^{\frac{1}{n}}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}), \quad \text{for } k = 0, \\
z_{(1)}^{\frac{1}{n}} &= r^{\frac{1}{n}}(\cos\frac{\theta + 2\pi}{n} + i\sin\frac{\theta + 2\pi}{n}), \quad \text{for } k = 1, \\
z_{(2)}^{\frac{1}{n}} &= r^{\frac{1}{n}}(\cos\frac{\theta + 4\pi}{n} + i\sin\frac{\theta + 4\pi}{n}), \quad \text{for } k = 2, \\
&\vdots \\
z_{(n-1)}^{\frac{1}{n}} &= r^{\frac{1}{n}}(\cos\frac{\theta + 2(n-1)\pi}{n} + i\sin\frac{\theta + 2(n-1)\pi}{n}), \\
&= r^{\frac{1}{n}}(\cos\frac{\theta - 2\pi}{n} + i\sin\frac{\theta - 2\pi}{n}), \quad \text{for} \quad k = (n-1). \qquad \blacksquare
\end{aligned}
\tag{78}
$$

13 EULER'S FORMULA

Taylor series[5] expansion of the trigonometric functions are:

$$
\begin{aligned}
\cos\theta &= 1 - \frac{1}{2!}\,\theta^2 + \frac{1}{4!}\,\theta^4 - \frac{1}{6!}\,\theta^6 + \cdots, \qquad \theta^2 < \infty, \\
\sin\theta &= \theta - \frac{1}{3!}\,\theta^3 + \frac{1}{5!}\,\theta^5 - \frac{1}{7!}\,\theta^7 + \cdots, \qquad \theta^2 < \infty.
\end{aligned}
$$

Using the algebraic properties of the imaginary number i

$$
i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \ldots, \quad (i)^{2n} = (-1)^n, \quad (i)^{2n+1} = (-1)^n i\ , \ \cdots.
$$

we can rewrite the power series representations in the form

$$
\begin{aligned}
\cos\theta &= 1 + \frac{1}{2!}\,(i\theta)^2 + \frac{1}{4!}\,(i\theta)^4 + \frac{1}{6!}\,(i\theta)^6 + \cdots + \frac{1}{(2n)!}(i\theta)^{2n} + \cdots, \\
i\sin\theta &= i\theta + \frac{1}{3!}\,(i\theta)^3 + \frac{1}{5!}\,(i\theta)^5 + \frac{1}{7!}\,(i\theta)^7 + \cdots + \frac{1}{(2n+1)!}(i\theta)^{2n+1} + \cdots.
\end{aligned}
$$

[5] Taylor, Brook (1685–1731).

Adding and subtracting these the two identities we find

$$\cos\theta \pm i\sin\theta = 1 \pm (i\theta) + \frac{1}{2!}\,(i\theta)^2 \pm \frac{1}{3!}\,(i\theta)^3 + \frac{1}{4!}\,(i\theta)^4 + \cdots + \frac{1}{2n!}\,(i\theta)^{2n} \pm \frac{1}{(2n+1)!}\,(i\theta)^{2n+1} + \cdots .$$

However, when $\theta^2 < \infty$, this infinite series in powers of $(i\theta)$ converges and it can be shown that it is the Taylor series expansion of the exponential function $\exp(\pm i\theta)$. Hence, we have a fundamental identity

$$\cos\theta \pm i\sin\theta \equiv \mathrm{e}^{\pm i\theta}, \qquad i = \sqrt{-1}, \qquad \blacksquare$$

which is called *Euler's Formula.* Here e is the Euler's constant[6] , defined as

$$\begin{aligned}
\mathrm{e} &= \lim_{n\to\infty}(1+\frac{1}{n})^n, \\
&= \lim_{n\to\infty}[1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}\,(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{3!}\,(\frac{1}{n})^3 + \cdots], \\
&= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!} + \cdots .
\end{aligned}$$

This infinite series converges and the value of Euler's constant is

$$e = 2.7\,1828\,1828\,45\,90\,45 \cdots . \qquad \diamond$$

In general for any complex number $z = (x + iy)$, we can write

$$\begin{aligned}
\mathrm{e}^z &= \mathrm{e}^{x+iy}, \\
&= \mathrm{e}^x\mathrm{e}^{iy}, \\
&= \mathrm{e}^x(\cos y + i\sin y); \\
\mathrm{e}^{\overline{z}} &= \mathrm{e}^{x-iy}, \\
&= \mathrm{e}^x\mathrm{e}^{-iy}, \\
&= \mathrm{e}^x(\cos y - i\sin y).
\end{aligned}$$

In particular, when $x = 0$, $\left.\mathrm{e}^x\right|_{x\to 0} = 1$ and we find

$$\mathrm{e}^{\pm iy} = \cos y \pm i\sin y,$$

and when $y = 0$ we get $\mathrm{e}^z = \mathrm{e}^x$.

We now show that complex numbers in polar form can be readily written in an exponential form with the use of Euler's formula. We thus write

$$\begin{aligned}
z &= x + iy, \\
&= r(\cos\theta + i\sin\theta) \quad = r\mathrm{e}^{i\theta} \quad = r\mathrm{e}^{i(\theta+2k\pi)}, \quad k = 0, 1, 2, \cdots, n, \cdots .
\end{aligned}$$

The first n roots of a complex number can now be written as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\mathrm{e}^{i(\frac{\theta+2k\pi}{n})}, \; k = 0, 1, 2, \ldots, (n-1).$$

We also note that

[6] See Appendix **B:** Euler's Constant e

i. For $k = n$

$$\begin{aligned} z^{\frac{1}{n}} &= r^{\frac{1}{n}} e^{i(\frac{\theta+2n\pi}{n})}, \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}+2i\pi}, \\ &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}}, \end{aligned}$$

because $e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1$. This value of $z^{\frac{1}{n}}$ for $k = n$ is exactly the value of $z^{\frac{1}{n}}$ for $k = 0$.

ii. For $k = n + 1$

$$\begin{aligned} z^{\frac{1}{n}} &= r^{\frac{1}{n}} e^{i(\frac{\theta+2(n+1)\pi}{n})}, \\ &= r^{\frac{1}{n}} e^{i\frac{\theta+2\pi}{n}+2i\pi}, \\ &= r^{\frac{1}{n}} e^{i\frac{\theta+2\pi}{n}}. \end{aligned}$$

This value of $z^{\frac{1}{n}}$ for $k = n + 1$ is exactly the value for $z^{\frac{1}{n}}$ for $k = 1$.

$\vdots$

$\vdots$

$\vdots$

iii. For $k = (2n - 1)$

$$\begin{aligned} z^{\frac{1}{n}} &= r^{\frac{1}{n}} e^{i(\frac{(\theta+2(2n-1))\pi}{n})}, \\ &= r^{\frac{1}{n}} e^{i(\frac{(\theta+2(n-1))\pi}{n})+2i\pi}, \\ &= r^{\frac{1}{n}} e^{i\frac{\theta+2(n-1)\pi}{n}}, \end{aligned}$$

which is the value of $z^{\frac{1}{n}}$ for $k = (n - 1)$.

If we continue this process we find that the n roots of the complex numbers z for $k = (0, 1, 2, \ldots, n - 1)$ are the same as those for $k = (n, n + 1, n + 2, \ldots, 2n - 1)$. The first n roots of a complex numbers can thus be identified with the sequence $k = (0, 1, 2, \ldots, n - 1)$. ■

14 THE n ROOTS OF UNITY

The solution of the equation $z^n = 1$, for positive values of the integer n are the n roots of unity. In polar form the equation $z = 1$ can be written as

$$z = \cos 2k\pi + i \sin 2k\pi = e^{2ik\pi}, \quad k = 0, 1, 2, \cdots .$$

Using DeMoivre's theorem we find

$$\begin{aligned} z^{\frac{1}{n}} &= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{n}} = (e^{2ik\pi})^{\frac{1}{n}}, \\ &= (\cos \frac{2k}{n}\pi + i \sin \frac{2k}{n}\pi) = e^{2i\frac{k}{n}\pi}, \quad k = 0, 1, 2, \ldots, (n - 1). \end{aligned}$$

If we write

$$\omega_k = e^{2i\frac{k\pi}{n}} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \ldots, (n - 1) ,$$

then

$$
\begin{aligned}
\omega_0 &= 1, \\
\omega_1 &= \mathrm{e}^{2i\pi/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \\
\omega_2 &= \mathrm{e}^{4i\pi/n} = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \\
&\vdots \\
\omega_{n-1} &= \mathrm{e}^{2i(1-\frac{1}{n})\pi} = \cos 2(1-\frac{1}{n})\pi + i\sin 2(1-\frac{1}{n})\pi, \\
&= \mathrm{e}^{-2i\pi/n} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n}.
\end{aligned}
$$

One can similarly show that

$$\omega_{n-2} = \mathrm{e}^{-4i\pi/n} = \cos\frac{4\pi}{n} - i\sin\frac{4\pi}{n},$$

and in general

$$\omega_{n-k} = \mathrm{e}^{-2i\frac{k\pi}{n}} = \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}.$$

From here one easily sees that

$$
\begin{aligned}
\omega_0 = \overline{\omega}_n, \quad \omega_1 = \overline{\omega}_{n-1}\ , \ \omega_2 = \overline{\omega}_{n-2}, \\
\omega_3 = \overline{\omega}_{n-3}\ , \quad \ldots\ , \ \omega_k = \overline{\omega}_{n-k},
\end{aligned}
\tag{79}
$$

and conversely $\omega_{n-k} = \overline{\omega}_k$. ■

15 GEOMETRIC REPRESENTATION OF THE ROOTS OF UNITY

We now show the roots of unity diagrammatically for $n = 2, 3, 4, 5, \ldots$

i. $n = 2$ The two roots are ω_0 and ω_1, where

$$
\begin{aligned}
\omega_0 &= \overline{\omega}_2 = 1, \\
\omega_1 &= \overline{\omega}_1 = -1.
\end{aligned}
$$

ii. $n = 3$ The three roots are ω_0, ω_1 and ω_2, where

$$
\begin{aligned}
\omega_0 &= \overline{\omega}_3 = 1, \\
\omega_1 &= \overline{\omega}_2 = -\frac{1}{2}(1 + i\sqrt{3}), \\
\omega_2 &= \overline{\omega}_1 = -\frac{1}{2}(1 - i\sqrt{3}).
\end{aligned}
$$

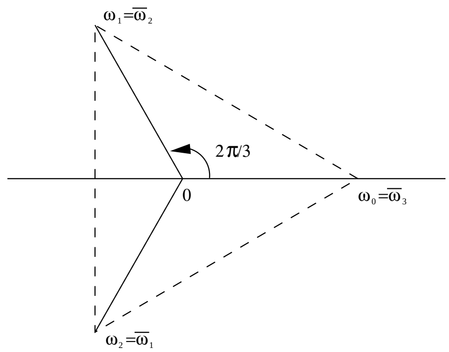

Figure 7.

iii. $n = 4$ The four roots are ω_0, ω_1, ω_2 and ω_3, where

$$\begin{aligned}
\omega_0 &= \overline{\omega}_4 = 1, \\
\omega_1 &= \overline{\omega}_3 = i, \\
\omega_2 &= \overline{\omega}_2 = -1, \\
\omega_3 &= \overline{\omega}_1 = -i.
\end{aligned}$$

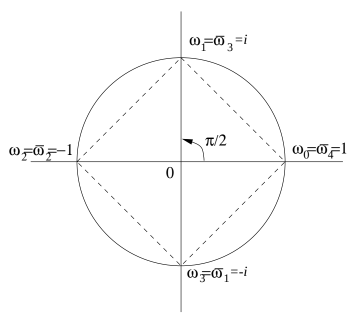

Figure 8.

iv. $n = 5$ The four roots are ω_0, ω_1, ω_2, ω_3 and ω_4, where

$$\begin{aligned}
\omega_0 &= \overline{\omega}_5 = 1, \\
\omega_1 &= \overline{\omega}_4 = (\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}), \\
\omega_2 &= \overline{\omega}_3 = (\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = -(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}), \\
\omega_3 &= \overline{\omega}_2 = (\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}) = -(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}), \\
\omega_4 &= \overline{\omega}_1 = (\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}) = (\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}).
\end{aligned}$$

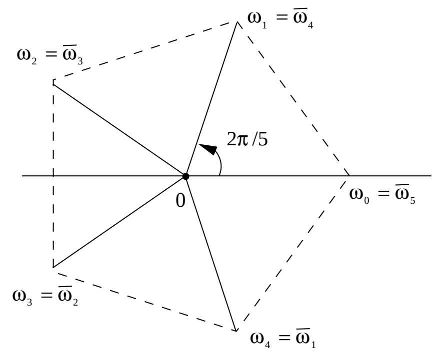

Figure 9.

Geometrically the n roots represent the n vertices of a regular polygon with n sides. This n-sided polygon is inscribed by a circle of radius $|z| = r = 1$, with center at the origin of the polygon. As an example the roots of the equation $z^6 = 1$ are shown by the vertices of a hexagon inscribed in a circle of radius $r = 1$ with center at the origin, as shown in the figure below.

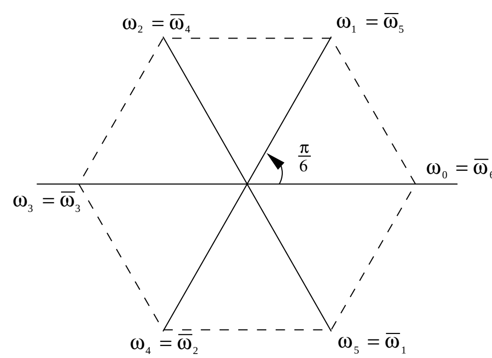

Figure 10.

16 EXAMPLES USING de MOIVRE'S THEOREM

In this section we prove de Moivre's theorem for complex numbers and then show some of its applications.

de Moivre's Theorem : For a uni–modular complex number

$$z = (\cos\theta + i\sin\theta),\ r = 1,\ 0 \leq \theta < 2\pi,$$

we have the identity

$$(\cos\theta + i\sin\theta)^n \equiv (\cos n\theta + i\sin n\theta),$$

for positive integer values of n.

PROOF : For a uni–modular complex number

$$z = (\cos\theta + i\sin\theta) = \mathrm{e}^{i\theta},$$

$$\begin{aligned} z^2 &= (\cos\theta + i\sin\theta)^2 = (\mathrm{e}^{i\theta})^2, \\ &= (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta, \\ &= (\cos 2\theta + i\sin 2\theta) = \mathrm{e}^{2i\theta}. \end{aligned}$$

We now assume that for some positive integer k

$$\begin{aligned} z^k &= (\cos\theta + i\sin\theta)^k = (\mathrm{e}^{i\theta})^k, \\ &= (\cos k\theta + i\sin k\theta) = \mathrm{e}^{ik\theta}. \end{aligned}$$

Then

$$\begin{aligned} z^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\mathrm{e}^{i\theta})^k\mathrm{e}^{i\theta}, \\ &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta), \\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta), \\ &= \{\cos[(k+1)\theta] + i\sin[(k+1)\theta]\} = \mathrm{e}^{i(k+1)\theta}. \end{aligned}$$

We have thus shown that if the result is true for some positive integer k, then it is also true for $(k+1)$. However, the result is true for $k = 1$, and therefore it is true for $k + 1 = 2$. By repeating the argument we conclude that if the result is true for $k = 2$, then it is also true for $k + 1 = 3$. By using this argument repeatedly we find that the result is true for all positive integer values of n. ■

We now show the use of de Moivre's theorem in developing some useful trigonometric identities.

17 TRIGONOMETRIC IDENTITIES USING de MOIVRE'S THEOREM

1. Prove the trigonometric identities

$$\begin{aligned} \cos 2\theta &\equiv (2\cos^2\theta - 1), \\ \frac{\sin 2\theta}{\sin\theta} &\equiv 2\cos\theta. \end{aligned}$$

To prove the identities we use de Moivre's theorem and write

$$\begin{aligned} \cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2, \\ &= (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta. \end{aligned}$$

Therefore

$$[\cos 2\theta - (\cos^2\theta - \sin^2\theta)] + i[\sin 2\theta - 2\sin\theta\cos\theta] \equiv 0.$$

For this complex valued identity to be true for all values of argument θ, its modulus must be zero. This requires that the real and imaginary parts be separately zero. Thus we have the identity

$$\begin{aligned} \cos 2\theta &= (\cos^2\theta - \sin^2\theta) = (2\cos^2\theta - 1), \\ \frac{\sin 2\theta}{\sin\theta} &= 2\cos\theta, \qquad \diamond \end{aligned}$$

where $(\sin^2\theta + \cos^2\theta) = 1$.

2. Prove the trigonometric identities

$$\frac{\cos 3\theta}{\cos\theta} = (4\cos^2\theta - 3),$$
$$\frac{\sin 3\theta}{\sin\theta} = (4\cos^2\theta - 1).$$

We use de Moivre's theorem and write

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3,\\ &= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3,\\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i\sin\theta(3\cos^2\theta - \sin^2\theta).\end{aligned}$$

Hence, we have the identity

$$[\cos 3\theta - (\cos^3\theta - 3\cos\theta\sin^2\theta)] + i[\sin 3\theta - \sin\theta(3\cos^2\theta - \sin^2\theta)] = 0.$$

For this identity to be true for all values of phase angle θ the modulus must be zero. Vanishing of the modulus requires that the real and imaginary parts of the complex number must each be zero. Hence, we have the idenitites

$$\cos 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta),$$
$$\frac{\sin 3\theta}{\sin\theta} = (3\cos^2\theta - \sin^2\theta).$$

Since $\sin^2\theta = 1 - \cos^2\theta$, these two identities can be rewritten as

$$\frac{\cos 3\theta}{\cos\theta} = (4\cos^2\theta - 3),$$
$$\frac{\sin 3\theta}{\sin\theta} = (4\cos^2\theta - 1). \qquad \diamond$$

3. Prove the trigonometric identities

$$\cos 4\theta = (8\cos^4\theta - 8\cos^2\theta + 1),$$
$$\frac{\sin 4\theta}{\sin\theta} = (8\cos^3\theta - 4\cos\theta).$$

To prove these identities we use de Moivre's theorem and write

$$\begin{aligned}(\cos 4\theta + i\sin 4\theta) &= (\cos\theta + i\sin\theta)^4,\\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i\sin\theta(4\cos^3\theta - 4\cos\theta\sin^2\theta),\\ &= (8\cos^4\theta - 8\cos^2\theta + 1) + i\sin\theta(8\cos^3\theta - 4\cos\theta).\end{aligned}$$

Hence, we have the identity

$$[\cos 4\theta - (8\cos^4\theta - 8\cos^2\theta + 1)] + i[\sin 4\theta - \sin\theta(8\cos^3\theta - 4\cos\theta)] = 0.$$

For this complex valued number to be zero for all values of phase angle θ, the modulus must be zero. Vanishing of the modulus requires that the real and imaginary parts of the complex number must each be zero. Hence, we have the two identities

$$\cos 4\theta = (8\cos^4\theta - 8\cos^2\theta + 1),$$
$$\frac{\sin 4\theta}{\sin\theta} = (8\cos^3\theta - 4\cos\theta). \qquad \diamond$$

4. Prove the trigonometric identities

$$\frac{\cos 5\theta}{\cos\theta} = (16\cos^4\theta - 20\cos^2\theta + 5),$$
$$\frac{\sin 5\theta}{\sin\theta} = (16\cos^4\theta - 12\cos^2\theta + 1).$$

To establish these identities we use de Moivre's theorem and binomial expansion. The binomial expansion we need is of the type [7]

$$\begin{aligned}(a+b)^5 &= \sum_{k=0}^{n} \binom{5}{k} a^{(5-k)} b^k, \\ &= \sum_{k=0}^{n} \binom{5}{k} b^{(5-k)} a^k, \\ &= a^5 + \binom{5}{1} a^4 b + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^3 + \binom{5}{4} a b^4 + b^5,\end{aligned}$$

where $\binom{m}{n} = \binom{m}{(m-n)} = \frac{m!}{(m-n)!n!}$ for $m > n \geq 0$. Hence

$$\begin{aligned}\binom{5}{0} &= \binom{5}{5} = \frac{5!}{5!0!} = 1, \\ \binom{5}{1} &= \binom{5}{4} = \frac{5!}{4!1!} = 5, \\ \binom{5}{2} &= \binom{5}{3} = \frac{5!}{3!2!} = 10,\end{aligned}$$

and therefore

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

We now use de Moivre's theorem and with the help of binomial expansion we find

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5, \\ &= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i\sin\theta(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta).\end{aligned}$$

Now

$$\begin{aligned}&\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &\qquad = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta), \\ &\qquad = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5),\end{aligned}$$

and

$$\begin{aligned}&5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta \\ &\qquad = 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1 - 2\cos^2\theta + \cos^4\theta), \\ &\qquad = 16\cos^4\theta - 12\cos^2\theta + 1.\end{aligned}$$

[7] This is a special case of the general theorem developed by Blaise Pascal [1623-1662] in the analysis of gambling problems.

Hence, suitably combining the two terms we have the identity

$$[\cos 5\theta - \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)] + i[\sin 5\theta - \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)] \equiv 0.$$

Using the familiar argument that for this identity to be true for all values of phase angle θ, the modulus must vanish and this leads us to the two identities

$$\begin{aligned}\frac{\cos 5\theta}{\cos\theta} &\equiv (16\cos^4\theta - 20\cos^2\theta + 5),\\ \frac{\sin 5\theta}{\sin\theta} &\equiv (16\cos^4\theta - 12\cos^2\theta + 1). \qquad \diamond\end{aligned}$$

These identities are true for all values of θ. As an example we consider limiting values of $\theta = 0$ and $\pi/2$. Then

(a)

$$\begin{aligned}\left.\frac{\cos 5\theta}{\cos\theta}\right]_{\theta=0} &= 1 = (16\cos^4\theta - 20\cos^2\theta + 5)_{\theta=0} = (16 - 20 + 5) = 1,\\ \left.\frac{\sin 5\theta}{\sin\theta}\right]_{\theta=0} &= 5\left.\frac{\cos 5\theta}{\cos\theta}\right]_{\theta=0} = 5 = (16\cos^4\theta - 12\cos^2\theta + 1)_{\theta=0} = (16 - 12 + 1) = 5.\end{aligned} \tag{80}$$

(b)

$$\begin{aligned}\left.\frac{\cos 5\theta}{\cos\theta}\right]_{\theta=\pi/2} &= \left.\frac{-5\sin 5\theta}{-\sin\theta}\right]_{\theta=\pi/2} = 5 = \ (16\cos^4\theta - 12\cos^2\theta + 5)_{\theta=\pi/2} = 5.\\ \left.\frac{\sin 5\theta}{\sin\theta}\right]_{\theta=\pi/2} &= 1 = (16\cos^4\theta - 12\cos^2\theta + 1)_{\theta=\pi/2} = 1.\end{aligned} \tag{81}$$

One can similarly show that

$$\begin{aligned}\cos 6\theta &\equiv (32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1),\\ \frac{\sin 6\theta}{\sin\theta} &\equiv (32\cos^5\theta - 32\cos^3\theta + 6\cos\theta). \qquad \diamond\end{aligned}$$

We next consider another set of useful trigonometric identities in which we express $\sin^n\theta$, $\cos^n\theta$ in terms of $\sin n\theta$ and $\cos n\theta$. We first write

$$\begin{aligned}\cos\theta &= \frac{1}{2}[(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)],\\ \sin\theta &= \frac{1}{2i}[(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)].\end{aligned}$$

5. Express $\cos^2\theta$ and $\sin^2\theta$ in terms of $\cos 2\theta$.

   We write

$$\cos^2\theta = \frac{1}{4}[(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)]^2,$$

and

$$\sin^2\theta = -\frac{1}{4}[(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)]^2.$$

Using DeMoivre's theorem to simplify these, we find

$$\begin{aligned}
\cos^2\theta &= \frac{1}{4}[(\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2 + 2],\\
&= \frac{1}{4}[(\cos 2\theta + i\sin 2\theta) + (\cos 2\theta - i\sin 2\theta) + 2],\\
&= \frac{1}{2}(1 + \cos 2\theta). \qquad\qquad \diamond\\
\sin^2\theta &= -\frac{1}{4}[(\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2 - 2],\\
&= -\frac{1}{4}[(\cos 2\theta + i\sin 2\theta) + (\cos 2\theta - i\sin 2\theta) - 2],\\
&= \frac{1}{2}(1 - \cos 2\theta). \qquad\qquad \diamond
\end{aligned}$$

6. Express $\cos^3\theta$ and $\sin^3\theta$ in terms of trigonometric functions of the type $\cos n\theta$ and $\sin n\theta$.

   We write

$$\begin{aligned}
\cos^3\theta = &\frac{1}{8}[(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)]^3,\\
= &\frac{1}{8}[(\cos\theta + i\sin\theta)^3 + 3(\cos\theta + i\sin\theta)^2(\cos\theta - i\sin\theta)\\
&+ 3(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)^2 + (\cos\theta - i\sin\theta)^3],\\
= &\frac{1}{8}[(\cos 3\theta + i\sin 3\theta) + 3(\cos 2\theta + i\sin 2\theta)(\cos\theta - i\sin\theta)\\
&+ 3(\cos\theta + i\sin\theta)(\cos 2\theta - i\sin 2\theta) + (\cos 3\theta - i\sin 3\theta)],\\
= &\frac{1}{8}[(\cos 3\theta + i\sin 3\theta) + 3(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta + i\sin 2\theta\cos\theta - i\cos 2\theta\sin\theta)\\
&+ 3(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta + i\sin\theta\cos 2\theta - i\cos\theta\sin 2\theta) + (\cos 3\theta - i\sin 3\theta)],\\
= &\frac{1}{4}(\cos 3\theta + 3\cos 2\theta\cos\theta + 3\sin 2\theta\sin\theta),\\
= &\frac{1}{4}[\cos 3\theta + 3\cos(2\theta - \theta)],\\
= &\frac{1}{4}(\cos 3\theta + 3\cos\theta). \qquad\qquad \diamond
\end{aligned}$$

$$\begin{aligned}
\sin^3\theta &= -\frac{1}{8i}[(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)]^3,\\
&= -\frac{1}{8i}[(\cos\theta + i\sin\theta)^3 - 3(\cos\theta + i\sin\theta)^2(\cos\theta - i\sin\theta)\\
&\quad + 3(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)^2 - (\cos\theta - i\sin\theta)^3],\\
&= -\frac{1}{8i}[(\cos 3\theta + i\sin 3\theta) - 3(\cos 2\theta + i\sin 2\theta)(\cos\theta - i\sin\theta)\\
&\quad + 3(\cos\theta + i\sin\theta)(\cos 2\theta - i\sin 2\theta) - (\cos 3\theta - i\sin 3\theta)],\\
&= -\frac{1}{8i}[(\cos 3\theta + i\sin 3\theta) - 3(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta + i\sin 2\theta\cos\theta - i\cos 2\theta\sin\theta)\\
&\quad + 3(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta + i\sin\theta\cos 2\theta - i\cos\theta\sin 2\theta) - (\cos 3\theta - i\sin 3\theta)],\\
&= -\frac{1}{4}[\sin 3\theta - 3(\cos\theta\sin 2\theta - \cos 2\theta\sin\theta)],\\
&= -\frac{1}{4}[\sin 3\theta - 3\sin(2\theta - \theta)],\\
&= \frac{1}{4}(3\sin\theta - \sin 3\theta). \qquad \diamond
\end{aligned}$$

7. Express $\cos^4\theta$ and $\sin^4\theta$ in terms of trigonometric functions of the type $\cos n\theta$ and $\sin n\theta$.

   We write

$$\begin{aligned}
\cos^4\theta &= \frac{1}{2^4}(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta})^4,\\
&= \frac{1}{2^4}(\mathrm{e}^{4i\theta} + 4\mathrm{e}^{3i\theta}\mathrm{e}^{-i\theta} + 6\mathrm{e}^{2i\theta}\mathrm{e}^{-2i\theta} + 4\mathrm{e}^{i\theta}\mathrm{e}^{-3i\theta} + \mathrm{e}^{-4i\theta}),\\
&= \frac{1}{2^4}[(\mathrm{e}^{4i\theta} + \mathrm{e}^{-4i\theta}) + 4(\mathrm{e}^{2i\theta} + \mathrm{e}^{-2i\theta}) + 6],\\
&= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3). \qquad \diamond
\end{aligned}$$

   Similarly we obtain

$$\begin{aligned}
\sin^4\theta &= \frac{1}{(2i)^4}(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta})^4,\\
&= \frac{1}{2^4}(\mathrm{e}^{4i\theta} - 4\mathrm{e}^{3i\theta}\mathrm{e}^{-i\theta} + 6\mathrm{e}^{2i\theta}\mathrm{e}^{-2i\theta} - 4\mathrm{e}^{i\theta}\mathrm{e}^{-3i\theta} + \mathrm{e}^{-4i\theta}),\\
&= \frac{1}{2^4}[(\mathrm{e}^{4i\theta} + \mathrm{e}^{-4i\theta}) - 4(\mathrm{e}^{2i\theta} + \mathrm{e}^{-2i\theta}) + 6],\\
&= \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3). \qquad \diamond
\end{aligned}$$

18 MISCELLANEOUS EXAMPLES SHOWING APPLICATIONS OF de MOIVRE'S THEOREM

1. Show that $(2i)^{1/2} = \pm(1 + i)$.

We write in polar form

$$2i = 2\mathrm{e}^{i(\frac{\pi}{2}+2k\pi)}, \text{ for } k = 0, 1, \cdots .$$

Then the two roots ($n = 2$) are

$$(2i)^{1/2} = \begin{cases} \sqrt{2}\mathrm{e}^{i(\frac{\pi/2+2k\pi}{2})} & k = 0, 1, \\ = \sqrt{2}\mathrm{e}^{i(\frac{\pi}{4}+k\pi)} & k = 0, 1. \end{cases}$$

Hence, the two roots are

$$(2i)^{1/2} = \begin{cases} \sqrt{2}\mathrm{e}^{i\frac{\pi}{4}} & \text{for } k = 0, \\ \sqrt{2}\mathrm{e}^{i(\frac{\pi}{4}+\pi)} & \text{for } k = 1. \end{cases}$$

However, $\mathrm{e}^{i\pi} = (\cos\pi + i\sin\pi) = -1$, and therefore the two roots can be written as

$$(2i)^{1/2} = \begin{cases} \sqrt{2}\mathrm{e}^{i\frac{\pi}{4}} & = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = (1+i), \\ -\sqrt{2}\mathrm{e}^{i\frac{\pi}{4}} & = -\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = -(1+i). \end{cases}$$

Hence $(2i)^{1/2} = \pm(1+i)$. $\diamond$

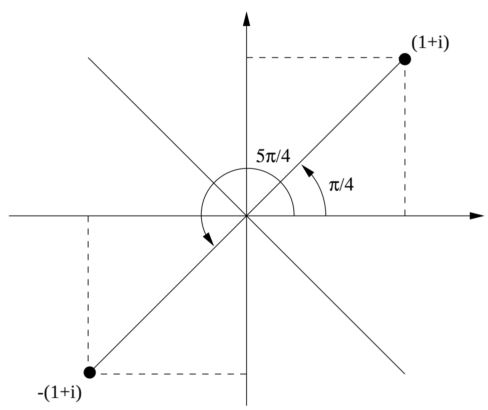

Figure 11.

2. Show that $z = (1+\sqrt{3}i)^{1/2} = \pm\frac{1}{\sqrt{2}}(\sqrt{3}+i)$.

   We will find the two roots using two different procedures.

   Method 1.

   We first write the complex number in polar form. Thus

$$\begin{aligned}(1+\sqrt{3}i) &= r\mathrm{e}^{i\theta}, \\ &= r\mathrm{e}^{i(\theta+2k\pi)}, \text{ for } \quad k = 0, 1, 2, \cdots\end{aligned}$$

   Here

$$\begin{aligned} r &= \sqrt{1+3} = 2, \\ \theta &= \tan^{-1}\sqrt{3} = \frac{\pi}{3}. \end{aligned}$$

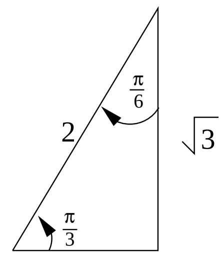

Figure 12.

Therefore

$$(1+\sqrt{3}i) = 2e^{i(\pi/3+2k\pi)}, \quad k = 0, 1, 2, \cdots .$$

The two roots ($n = 2$) are

$$(1+\sqrt{3}i)^{1/2} = \sqrt{2}e^{i(\pi/6+k\pi)}, \quad k = 0, 1.$$

Explicitly, the two roots take the form

$$\begin{aligned} z_0 &= \sqrt{2}e^{i\pi/6} = \sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}), \\ &= \sqrt{2}(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \frac{1}{\sqrt{2}}(\sqrt{3}+i), \text{ for } \quad k = 0; \\ z_1 &= \sqrt{2}e^{(i\pi/6+\pi)} = -\sqrt{2}e^{i\pi/6} = -\frac{1}{\sqrt{2}}(\sqrt{3}+i), \quad \text{for } \; k = 1. \end{aligned}$$

Hence, the two distinct roots of the complex number z are

$$\begin{matrix} z_0 \\ z_1 \end{matrix} = \pm\frac{1}{\sqrt{2}}(\sqrt{3}+i). \qquad \diamond$$

These two roots are shown diagramatically in the figure below.

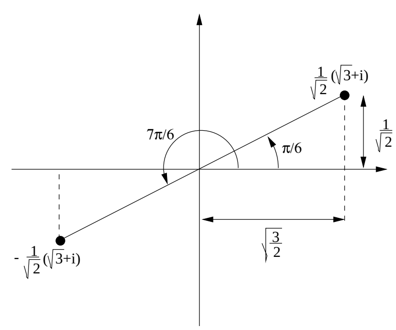

Figure 13.

We now determine the polynomial equation whose two roots are z_0 and z_1. Such a polynomial equation is

$$(z-z_0)(z-z_1) = z^2 - \mathbf{I}z + \mathbf{II} = 0.$$

The first and second invariants of this polynomial are

$$\mathbf{I:}\quad z_0 + z_1 \equiv 0 \qquad\qquad \text{because } z_1 = -z_0;$$
$$\mathbf{II:}\quad z_0 z_1 = \frac{1}{\sqrt{2}}(\sqrt{3}+i) \times (-\frac{1}{\sqrt{2}})(\sqrt{3}+i) = -(1+i\sqrt{3}).$$

From here we conclude that z_0 and z_1 are the roots of the second order polynomial equation

$$z^2 - (1+i\sqrt{3}) = 0. \qquad \diamond$$

Method 2.

To find the two roots of a complex number we write

$$z = (1+\sqrt{3}i)^{1/2} = x + iy\ .$$

We now attempt to find x and y without converting the complex number into its polar representation. Squaring both sides of the equation we get

$$1 + \sqrt{3}i = (x^2 - y^2) + 2ixy\ .$$

For this equation to be true for all values of x and y, the real and imaginary parts of the equation must be equal. Thus we have two equations

$$x^2 - y^2 = 1\ , \tag{82}$$
$$2xy = \sqrt{3}\ . \tag{83}$$

We can eliminate the variable y from these two equations. Thus from the second equation we find $y = \sqrt{3}/2x$ and substituting it in the first equation we get the equation

$$x^2 - (\sqrt{3}/2x)^2 = 1\ ,$$

or equivalently

$$x^4 - x^2 - \frac{3}{4} = 0\ ,$$

which is a bi-quadratic equation in the remaining variable x. The two roots of the bi-quadratic equation are

$$(x_1^2;\ x_2^2) = \frac{1}{2}(1 \pm \sqrt{1+3}) = \left(\frac{3}{2}\ ;\ -\frac{1}{2}\right)\ .$$

Or, the four roots are

$$(x_1, x_2;\ x_3, x_4) = \left(\sqrt{\frac{3}{2}}\ ,\ -\sqrt{\frac{3}{2}}\ ;\ \frac{1}{\sqrt{2}}i\ ,\ -\frac{1}{\sqrt{2}}i\right)\ . \qquad \diamond$$

Since $y = \sqrt{3}/2x$, the corresponding four values of y are

$$y_1 = \frac{\sqrt{3}}{2}\sqrt{\frac{2}{3}} \quad = \frac{1}{\sqrt{2}}\ ,$$
$$y_2 = -\frac{\sqrt{3}}{2}\sqrt{\frac{2}{3}} = -\frac{1}{\sqrt{2}}\ ;$$
$$y_3 = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{i} \quad = -i\sqrt{\frac{3}{2}}\ ,$$
$$y_4 = -\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{i} = i\sqrt{\frac{3}{2}}\ . \qquad \diamond$$

Hence

	(1)	(2)	(3)	(4)
$x =$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}i$	$-\frac{1}{\sqrt{2}}i$
$y =$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-i\sqrt{\frac{3}{2}}$	$i\sqrt{\frac{3}{2}}$
$x + iy =$	$\frac{(\sqrt{3}+i)}{\sqrt{2}}$	$-\frac{(\sqrt{3}+i)}{\sqrt{2}}$	$\frac{(\sqrt{3}+i)}{\sqrt{2}}$	$-\frac{(\sqrt{3}+i)}{\sqrt{2}}$

Note that the two roots in the third and fourth columns are repetition of the roots in the first and second columns. Since our polynomial equation is a quadratic equation, only the two roots in the first and second columns are valid roots.

3. Find the three roots of the cubic equation

$$z^3 + 1 = 0.$$

Here $z = (-1)^{1/3}$ and we first write it in polar form. Then

$$z_k = \mathrm{e}^{i(\frac{\pi+2k\pi}{3})} \qquad \text{for } k = 0, 1 \text{ and } 2.$$

Hence, the three roots are

$$\begin{aligned} z_0 &= \mathrm{e}^{i\pi/3} = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \frac{1}{2}(1 + i\sqrt{3}) \ \text{, for } \ k = 0, \\ &= \mathrm{e}^{\frac{\pi+2\pi}{3}} = \mathrm{e}^{i\pi} = (\cos\pi + i\sin\pi) = -1 \ \text{, for } \ k = 1, \\ &= \mathrm{e}^{i(\frac{\pi+4\pi}{3})} = \mathrm{e}^{-i\pi/3} = (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = \frac{1}{2}(1 - i\sqrt{3}) \ \text{, for } \ k = 2. \end{aligned} \tag{84}$$

Thus the three roots for $k = 0, 1$ and 2, are

$$\begin{aligned} z_0 &= \frac{1}{2}(1 + i\sqrt{3}), \\ z_1 &= -1, \qquad\qquad \diamond \\ z_2 &= \frac{1}{2}(1 - i\sqrt{3}). \end{aligned}$$

Note that of the three roots, the root z_1 is real; the remaining two roots form a complex conjugate pair.

The three invariants of a cubic polynomial equation are

$$\begin{aligned} \mathbf{I:} &\quad z_0 + z_1 + z_2 = \frac{1}{2}(1 + i\sqrt{3}) - 1 + \frac{1}{2}(1 - i\sqrt{3}) \equiv 0. \\ \mathbf{II:} &\quad z_0z_1 + z_1z_2 + z_2z_0 = \frac{1}{2}(1 + i\sqrt{3})(-1) + (-1)\frac{1}{2}(1 - i\sqrt{3}) + \frac{1}{4}(1 + i\sqrt{3})(1 - i\sqrt{3}) \equiv 0. \\ \mathbf{III:} &\quad z_0z_1z_2 = \frac{1}{2}(1 + i\sqrt{3})(-1)\frac{1}{2}(1 - i\sqrt{3}) \equiv -1. \end{aligned} \tag{85}$$

Hence, the three roots $\{z^0, z^1, z^2\}$ are the zeros of the polynomial equation

$$(z - z_0)(z - z_1)(z - z_2) = z^3 - \mathbf{I}z^2 + \mathbf{II}z - \mathbf{III}, \\ = z^3 + 1, \qquad \diamond$$

where the first and second invariants are identically zero [8].

The three roots are shown diagramatically in the figure below.

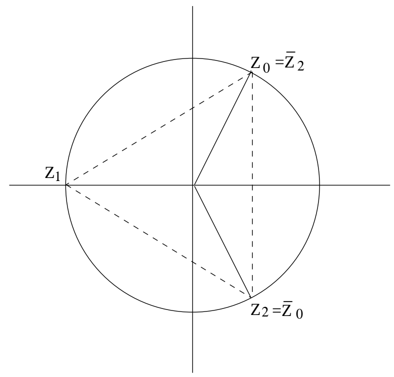

Figure 14.

4. Evaluate $(-16)^{1/4}$ and verify the result.

Using polar form

$$-16 = 16\mathrm{e}^{i(\pi+2k\pi)},\ k = 0, 1, 2, \cdots.$$

Therefore

$$z_k = (-16)^{1/4} = 2\mathrm{e}^{i(\frac{\pi+2k\pi}{4})},\ k = 0, 1, 2, 3.$$

Hence, the four roots are

(a)

$$z_0 = 2\mathrm{e}^{i\pi/4} = 2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \qquad = \sqrt{2}(1+i).$$

(b)

$$z_1 = 2\mathrm{e}^{i3\pi/4} = 2\mathrm{e}^{i(\pi-\frac{\pi}{4})}, \\ = -2\mathrm{e}^{-i\pi/4} = -2(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = -\sqrt{2}(1-i).$$

(c)

$$z_2 = 2\mathrm{e}^{i5\pi/4} = 2\mathrm{e}^{i(\pi+\frac{\pi}{4})}, \\ = -2\mathrm{e}^{i\pi/4} = -2(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = -\sqrt{2}(1+i).$$

[8] See Appendix **B** for invariants of a polynomial equation.

(d)

$$\begin{aligned} z_3 &= 2e^{i7\pi/4} = 2e^{i(2\pi - \frac{\pi}{4})}, \\ &= 2e^{-i\pi/4} = 2(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = \sqrt{2}(1-i). \end{aligned}$$

Hence, expressing succinctly the four roots are

$$\begin{matrix} z_0 \\ z_3 \end{matrix} = \sqrt{2}(1 \pm i), \qquad \begin{matrix} z_1 \\ z_2 \end{matrix} = -\sqrt{2}(1 \mp i). \qquad \diamond$$

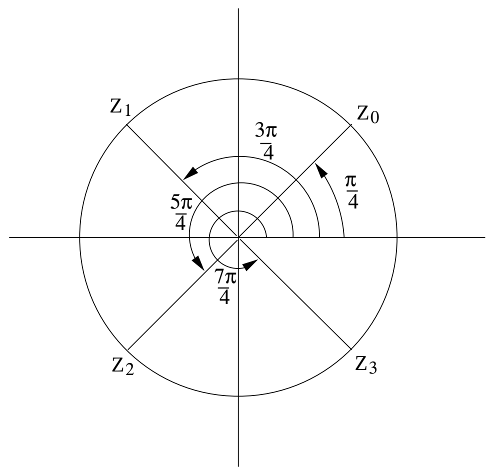

Figure 15.

Note that $z_3 = \overline{z}_0$ and $z_2 = \overline{z}_1$. Explicitly, these roots are

$$\begin{aligned} z_0 &= \sqrt{2}(1+i), \\ z_1 &= \sqrt{2}(-1+i), \\ z_2 &= \sqrt{2}(-1-i), \\ z_3 &= \sqrt{2}(1-i). \end{aligned}$$

The four invariants are

$$\begin{aligned} \mathbf{I:} \quad & (z_0 + z_3) + (z_1 + z_2) = \sqrt{2}[(1+i) + (1-i) + (-1+i) + (-1-i)] \equiv 0, \\ \mathbf{II:} \quad & (z_0 z_1 + z_0 z_2 + z_0 z_3 + z_1 z_2 + z_1 z_3 + z_2 z_3), \\ & = 2[-(1+i)(1-i) - (1+i)^2 + (1+i)(1-i) + (1-i)(1+i) - (1-i)^2 - (1-i)(1+i)], \\ & = 2[-2 - 2i + 2 + 2 + 2i - 2] \equiv 0. \\ \mathbf{III:} \quad & (z_1 z_2 z_3 + z_0 z_2 z_3 + z_0 z_1 z_3 + z_0 z_1 z_2) \equiv 0. \\ \mathbf{IV:} \quad & (z_0 z_1 z_2 z_3) = 4[(1+i)^2(1-i)^2] = 4 \times 4 = 16. \end{aligned} \tag{87}$$

Hence the four roots $\{z_0, z_1, z_2, z_3\}$ are the roots of the bi-quadratic polynomial equation $z^4 + 16 = 0$. $\diamond$

5. Find the roots of the complex number $z = (5+12i)^{1/2}$.

We assume that the roots of the complex number can be written as

$$(5+12i)^{1/2} = u + iv.$$

Squaring both sides of this equation we get

$$5 + 12i = (u^2 - v^2) + 2iuv.$$

Equating real and imaginary parts of this equation, we obtain two equations in two unknowns u and v. These two equations are

$$\begin{aligned} u^2 - v^2 &= 5, \\ uv &= 6. \end{aligned}$$

Using these two equations, we first find $(u^2 + v^2)$. Thus

$$\begin{aligned} u^2 + v^2 &= \sqrt{(u^2 - v^2)^2 + 4u^2v^2}, \\ &= \sqrt{25 + 144} = 13. \end{aligned}$$

Hence, we have a set of two linearly independent simultaneous equations

$$\begin{aligned} u^2 - v^2 &= 5, \\ u^2 + v^2 &= 13. \end{aligned}$$

The non–trivial solution is

$$u^2 = 9,\ v^2 = 4.$$

Hence

$$\begin{aligned} u &= \pm 3, \\ v &= \pm 2. \end{aligned}$$

The roots are

$$\begin{matrix} z_0 \\ z_1 \end{matrix} = \pm(3+2i); \quad \begin{matrix} \overline{z}_0 \\ \overline{z}_1 \end{matrix} = \pm(3-2i). \qquad \diamond$$

Here, z_0 and z_1 are the two roots of $(5+12i)^{1/2}$ and $\overline{z}_0$ and $\overline{z}_1$ are the two roots of $(5-12i)^{1/2}$.

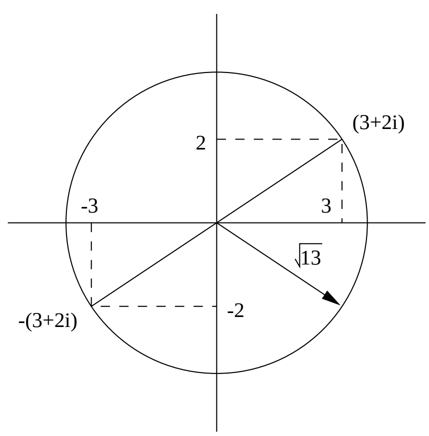

Figure 16.

19 COMPLEX NUMBER AND ITS INVERSE

Corresponding to every complex number

$$z = x + iy,$$

there exists the inverse complex number

$$z^{-1} = u + iv,$$

such that

$$zz^{-1} = z^{-1}z = 1.$$

Forming the product zz^{-1} we find

$$\begin{aligned} zz^{-1} &= (x+iy)(u+iv), \\ &= (xu - yv) + i(xv + yu) = 1. \end{aligned}$$

Equating real and imaginary parts we get a pair of simultaneous equations

$$\begin{aligned} xu - yv &= 1, \\ yu + xv &= 0. \end{aligned}$$

The linearly independent unknown variables are u and v, and x and y are the two given coefficients. The determinant of the coefficients of the two independent variables in these two simultaneous equations is

$$\Delta = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0,$$

because the modulus of the given complex number is non-zero. Therefore these two equations are linearly independent and a unique non-trivial solution exists.

To apply Cramer's rule for determining the solution (u, v), we write the two equations in matrix form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Using Cramer's rule the solution is

$$\begin{aligned} u &= \frac{\begin{vmatrix} 1 & -y \\ 0 & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{x}{x^2+y^2}, \quad (x,y) \neq (0,0), \\ v &= \frac{\begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{-y}{x^2+y^2}, \quad (x,y) \neq (0,0). \end{aligned} \tag{88}$$

Therefore the inverse is

$$\begin{aligned} u + iv &= \frac{x - iy}{x^2+y^2}, \\ &= \frac{x - iy}{(x+iy)(x-iy)} = \frac{1}{x+iy}, \quad (x,y) \neq (0,0). \end{aligned}$$

Hence, given a complex number $z = x + iy$, the inverse complex number z^{-1} exists if the Jacobean $\Delta \neq 0$; in which case the inverse is given by

$$z^{-1} = u + iv = \frac{1}{x + iy}, \qquad (x, y) \neq (0, 0).$$

Example. Given $z = (2 + 3i)$. Find z^{-1} such that $zz^{-1} = 1$.

Let $z^{-1} = (u + iv)$. Then

$$\begin{aligned} zz^{-1} &= (2 + 3i)(u + iv), \\ &= (2u - 3v) + i(2v + 3u) = 1. \end{aligned}$$

Equating real and imaginary parts we get two equations

$$\begin{aligned} 2u - 3v &= 1, \\ 3u + 2v &= 0. \end{aligned}$$

The determinant of the coefficients of these two equations is

$$\Delta \equiv \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13 \neq 0.$$

Since $\Delta \neq 0$, the two equations are linearly independent and a unique inverse exists. Using Cramer's rule the solution is:

$$\begin{aligned} u &= \frac{\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix}}{\Delta} = \frac{2}{13}, \\ v &= \frac{\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}}{\Delta} = -\frac{3}{13}. \end{aligned}$$

Hence, the inverse is

$$z^{-1} = (u + iv) = \frac{1}{13}(2 - 3i). \qquad \diamond$$

20 AN IDENTITY INVOLVING ARGUMENTS

In this section we deal with the phase of the product and quotient of two complex numbers. We first prove a simple

THEOREM. Given two complex numbers z_1 and z_2

$$\begin{aligned} \arg(z_1 z_2) &= (\arg z_1 + \arg z_2)\ , \\ \arg(\frac{z_1}{z_2}) &= (\arg z_1 - \arg z_2). \end{aligned}$$

PROOF. We write the two complex numbers in polar form, i.e.,

$$z_1 = r_1 e^{i\theta_1}, \quad 0 \le \theta_1 < 2\pi,$$
$$z_2 = r_2 e^{i\theta_2}, \quad 0 \le \theta_2 < 2\pi.$$

Then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)},$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Thus the arguments

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) = (\arg z_1 + \arg z_2)\ ,$$
$$\arg(\frac{z_1}{z_2}) = (\theta_1 - \theta_2) = (\arg z_1 - \arg z_2). \qquad \blacksquare$$

Example 1. Given two complex numbers

$$z_1 = -1,\ z_2 = i,$$

find the arguments of the product fuction $z_1 z_2$ and of the quotient function z_1/z_2.

We write these functions in polar form. Then

$$z_1 = -1 = e^{i\pi},$$
$$z_2 = i = e^{i\pi/2}.$$

Therefore

$$z_1 z_2 = e^{i3\pi/2} = -i,$$
$$\frac{z_1}{z_2} = e^{i\pi/2} = \quad i.$$

Hence

$$\arg(z_1 z_2) = (\theta_1 + \theta_2) = \frac{3\pi}{2},$$
$$\arg(\frac{z_1}{z_2}) = (\theta_1 - \theta_2) = \frac{\pi}{2}. \qquad \diamond$$

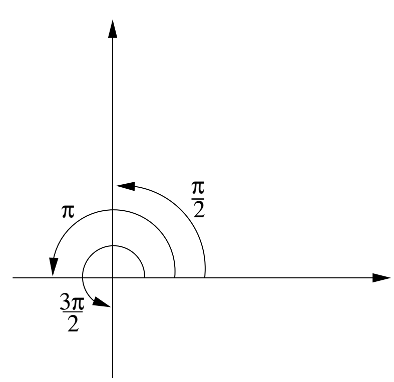

Figure 17.

Example 2. Find the phase θ when $z = -\dfrac{2}{(1+i\sqrt{3})}$.

We write

$$\begin{aligned} z &= -\frac{2}{(1+i\sqrt{3})}, \\ &= \frac{2}{r}e^{-i\theta}, \end{aligned}$$

where $r = \sqrt{1+3} = 2$, $\theta = \tan^{-1}\sqrt{3} = \pi/3$. Hence, in polar form

$$\begin{aligned} z &= -\frac{2}{2}e^{-i\pi/3} = -e^{-i\pi/3}, \\ &= e^{i\pi - i\pi/3} = e^{2i\pi/3}. \end{aligned}$$

Therefore the phase angle θ is $2\pi/3$ when measured positive in the anticlockwise sense. Also note that the complex number z can be written as

$$\begin{aligned} z &= -e^{-i\pi/3} = -(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}), \\ &= -\frac{1}{2}(1 - i\sqrt{3}). \qquad \diamond \end{aligned}$$

Example 3. Find the phase θ when $z = -\dfrac{i}{2(1+i)}$.

Let $z_1 = -i$ and $z_2 = (1+i)$. In polar form these two complex numbers have the representation

$$\begin{aligned} z_1 &= -i = e^{-i\pi/2}, \\ z_2 &= 1 + i = \sqrt{2}e^{i\theta}, \text{ where } \theta = \tan^{-1}1 = \frac{\pi}{4}, \\ &= \sqrt{2}e^{i\pi/4}. \end{aligned}$$

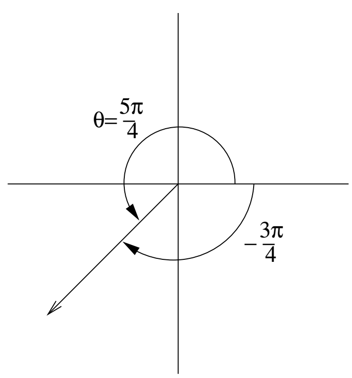

Figure 18.

Therefore

$$\begin{aligned} z &= -\frac{1}{2}\,\frac{i}{(1+i)} = \frac{1}{2\sqrt{2}}e^{-i\frac{\pi}{2} - i\frac{\pi}{4}}, \\ &= \frac{1}{2\sqrt{2}}e^{-i\frac{3\pi}{4}} = \frac{1}{2\sqrt{2}}e^{i\frac{5\pi}{4}}. \end{aligned}$$

Hence, the phase angle measured positive counterclockwise is $\theta = (5/4)\pi$. $\diamond$

Also note that the complex number z can be written as

$$z = \frac{1}{2\sqrt{2}} e^{i\pi + i\pi/4} = -\frac{1}{4}(1+i). \qquad \diamond \tag{90}$$

21 THE ROOTS OF A COMPLEX VALUED EQUATION

Example 1. Find the values of x and y such that

$$e^z = (1+i),$$

where $z = (x+iy)$.

Using Euler's formula we write

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) = (1+i),$$

or

$$(e^x \cos y - 1) + i(e^x \sin y - 1) = 0.$$

For this to be an identity for all values of x and y, the real and imaginary parts must be zero. Thus we have two equations

$$e^x \cos y = 1,$$
$$e^x \sin y = 1.$$

Differentiating these two equations we get a system of homogeneous equations

$$e^x \cos y \, dx - e^x \sin y \, dy = 0,$$
$$e^x \sin y \, dx + e^x \cos y \, dy = 0.$$

The determinant of the coefficients dx and dy is

$$\Delta = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0, \qquad \text{for } -\infty < x < \infty.$$

Hence the solution exists.

From the two equations we obtain upon division

$$\tan y = 1.$$

Next, take the square of each equation and add the two. We obtain

$$e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} = 2.$$

We thus have two uncoupled equations

$$e^{2x} = 2,$$
$$\tan y = 1,$$

and we have to find the roots of these two equations.

1. $\tan y = 1$.
   Let the solution be $y = \frac{\pi}{4} + \theta_n$. Then

$$\tan y = \tan(\frac{\pi}{4} + \theta_n) = \frac{1 + \tan\theta_n}{1 - \tan\theta_n} = 1.$$

   Hence,

$$1 + \tan\theta_n = 1 - \tan\theta_n,$$

   or,

$$\tan\theta_n = 0.$$

   Hence, the unknown θ_n is the solution of the transcedental equation. Its roots are

$$\theta_n = 2n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots \pm \cdots.$$

   Therefore

$$y_n = \frac{\pi}{4} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \cdots \pm \cdots.$$

2. $e^{2x} = 2$.
   We write this equation as

$$e^{2x} = e^{\ln 2 + 2im\pi}, \qquad m = 0, \pm 1, \pm 2, \cdots \pm \cdots.$$

   The complex roots of this equation are

$$\begin{aligned} x_m &= \frac{1}{2}\ln 2 + im\pi, \\ &= \ln\sqrt{2} + im\pi, \quad m = 0, \pm 1, \pm 2, \cdots \pm \cdots. \end{aligned}$$

Hence x and y have the values

$$\begin{aligned} x_m &= \ln\sqrt{2} + im\pi, & m &= 0, \pm 1, \pm 2, \cdots \pm \cdots, \\ y_n &= \frac{\pi}{4} + 2n\pi, & n &= 0, \pm 1, \pm 2, \cdots \pm \cdots; \end{aligned}$$

or

$$z_{mn} = x_m + iy_n = \ln\sqrt{2} + i(1/4 + 2n + m)\pi, \quad m, n = 0, \pm 1, \pm 2, \pm \cdots. \qquad \diamond$$

22 INVARIANTS OF A POLYNOMIAL

According to *the fundamental theorem of linear algebra*, an n-th order polynomial in terms of the the complex variable z has n roots. Some of these roots may be real and the other complex. The complex roots occur in pairs as complex conjugates.

1. Consider a second order polynomial P_2. Let the two zeros of this polynomial be z_0 and z_1. Then the quadratic polynomial can be written in the factored form

$$P_2 = (z - z_0)(z - z_1).$$

This polynomial of second degree can also be written as

$$P_2 = z^2 - \mathbf{I}z + \mathbf{II}.$$

Here, **I** and **II** are the first and the second invariants of the quadratic polynomial and their values are

$$\begin{aligned}\mathbf{I} &= (z_0 + z_1),\\ \mathbf{II} &= z_0 z_1.\end{aligned}$$

These quantities **I** and **II** are called invariants because their value does not depend upon the choice of coordinates and once these two invariants are known, the polynomial P_2 is known.

As an example consider a second order polynomial

$$P_2 = (i - 2)(i - 3).$$

The invariants of this polynomial are

$$\begin{aligned}\mathbf{I} &= 2 + 3 = 5,\\ \mathbf{II} &= 2 \times 3 = 6.\end{aligned}$$

Hence, the polynomial P_2 is

$$\begin{aligned}P_2 &= i^2 - \mathbf{I}i + \mathbf{II} = -1 - 5i + 6,\\ &= 5(1 - i).\end{aligned}$$

2. Consider a cubic polynomial P_3 with three distinct zeros z_0, z_1 and z_2. Then according to the *fundamental theorem of algebra* we can write the cubic in the factored form

$$\begin{aligned}P_3 &= (z - z_0)(z - z_1)(z - z_2),\\ &= z^3 - \mathbf{I}z^2 + \mathbf{II}z - \mathbf{III}.\end{aligned}$$

Here **I**, **II** and **III** are the first, second and third invariants of a cubic polynomial and their values are

$$\begin{aligned}\mathbf{I} &= z_0 + z_1 + z_2,\\ \mathbf{II} &= z_0 z_1 + z_1 z_2 + z_2 z_0,\\ \mathbf{III} &= z_0 z_1 z_2.\end{aligned}$$

As an example consider a cubic polynomial

$$P_3 = (i - 1)(i - 2)(i - 3).$$

The three invariants are

$$\begin{aligned}\mathbf{I} &= 1 + 2 + 3 = 6,\\ \mathbf{II} &= 1 \times 2 + 1 \times 3 + 2 \times 3 = 11,\\ \mathbf{III} &= 1 \times 2 \times 3 = 6.\end{aligned}$$

Hence, the cubic polynomial can be written as

$$\begin{aligned}P_3 &= i^3 - 6i^2 + 11i - 6,\\ &= -i + 6 + 11i - 6 = 10i.\end{aligned}$$

3. Consider now a quadric polynomial with four distinct zeros z_0, z_1, z_2 and z_3. Using the fundamental theorem of algebra the quadric polynomial can be written in the factored form

$$P_4 = (z - z_0)(z - z_1)(z - z_2)(z - z_3).$$

This can be rewritten as

$$\begin{aligned} P_4 &= P_3(z - z_3), \\ &= (z^3 - \mathbf{I}_c z^2 + \mathbf{II}_c z - \mathbf{III}_c)(z - z_3), \end{aligned}$$

where $\mathbf{I}_c$, $\mathbf{II}_c$ and $\mathbf{III}_c$ are the invariants of a cubic polynomial. Simplifying it takes the form

$$\begin{aligned} P_4 &= (z^3 - \mathbf{I}_c z^2 + \mathbf{II}_c z - \mathbf{III}_c)(z - z_3), \\ &= z^4 - (\mathbf{I}_c + z_3)z^3 + (\mathbf{II}_c + \mathbf{I}_c z_3)z^2 - (\mathbf{III}_c + \mathbf{II}_c z_3)z + \mathbf{III}_c z_3, \\ &= z^4 - \mathbf{I}z^3 + \mathbf{II}z^2 - \mathbf{III}z + \mathbf{IV}, \end{aligned}$$

where the four fundamental invariants are

$$\begin{aligned} \mathbf{I} &= \mathbf{I}_c + z_3, \\ \mathbf{II} &= \mathbf{II}_c + \mathbf{I}_c z_3, \\ \mathbf{III} &= \mathbf{III}_c + \mathbf{II}_c z_3, \\ \mathbf{IV} &= \mathbf{III}_c z_3. \end{aligned}$$

Simplifying, these four invariants take the form:

$$\begin{aligned} \mathbf{I} &= z_0 + z_1 + z_2 + z_3, \\ \mathbf{II} &= (z_0 z_1 + z_0 z_2 + z_0 z_3) + (z_1 z_2 + z_1 z_3) + z_2 z_3, \\ \mathbf{III} &= z_0 z_1 z_2 + z_0 z_1 z_3 + z_0 z_2 z_3 + z_1 z_2 z_3, \\ \mathbf{IV} &= z_0 z_1 z_2 z_3. \end{aligned}$$

(a) As an example consider the quadric polynomial

$$P_4 = (i - 1)^4.$$

Here, $z = i$ and $z_0 = z_1 = z_2 = z_3 = 1$. Hence the four invariants are

$$\begin{aligned} \mathbf{I} &= 1 + 1 + 1 + 1 = 4, \\ \mathbf{II} &= (1 \times 1 + 1 \times 1 + 1 \times 1) + (1 \times 1 + 1 \times 1) + 1 \times 1 = 6, \\ \mathbf{III} &= 1 \times 1 \times 1 + 1 \times 1 \times 1 + 1 \times 1 \times 1 + 1 \times 1 \times 1 = 4, \\ \mathbf{IV} &= 1 \times 1 \times 1 \times 1 = 1. \end{aligned}$$

Hence the polynomial reduces to

$$\begin{aligned} P_4 &= (i)^4 - \mathbf{I}(i)^3 + \mathbf{II}(i)^2 - \mathbf{III}(i) + \mathbf{IV}, \\ &= 1 + 4i - 6 - 4i + 1 = -4. \end{aligned}$$

(b) As another example, using the properties of invariants we evaluate the quotient polynomial

$$Q = \frac{P_3}{P_2} = \frac{(i+1)(i+2)(i+3)}{(i-1)^2},$$

where P_3 is a cubic polynomial and P_2 is a quadratic polynomial.

For P_3 the three invariants are

$$\begin{aligned}\mathbf{I} &= -1-2-3=-6,\\ \mathbf{II} &= (-1\times -2)+(-1\times -3)+(-2\times -3)=11,\\ \mathbf{III} &= (-1)\times(-2)\times(-3)=-6.\end{aligned}$$

Hence the cubic polynomial is

$$\begin{aligned}P_3 &= (i)^3+6(i)^2+11(i)+6,\\ &= -i-6+11i+6=10i.\end{aligned}$$

For the quadratic polynomial P_2 the two invariants are

$$\begin{aligned}\mathbf{I} &= 1+1=2,\\ \mathbf{II} &= 1\times 1=1.\end{aligned}$$

Hence the polynomial P_2 reduces to the form

$$\begin{aligned}P_2 &= (i)^2-2i+1,\\ &= -2i.\end{aligned}$$

Hence, the quotient is

$$Q=\frac{P_3}{P_2}=\frac{10i}{-2i}=-5.$$

(c) As another example we consider the quadratic

$$P_4=(1-i)^4.$$

We will evaluate this using two different methods:

i. Here $z=-i$ and $z_0=z_1=z_2=z_3=1$. The four invariants are

$$\mathbf{I}=4,\ \mathbf{II}=6,\ \mathbf{III}=4,\ \mathbf{IV}=1.$$

Hence

$$\begin{aligned}P_4 &= (-i)^4+4(-i)^3+6(-i)^2+4(-i)+1,\\ &= 1+4i-6-4i+1=-4.\end{aligned}$$

ii. The modulus of the complex number $(1-i)$ is $\sqrt{2}$, and therefore we write

$$\begin{aligned}1-i &= \sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right),\\ &= \sqrt{2}(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4})=\sqrt{2}\mathrm{e}^{-i\pi/4}.\end{aligned}$$

Hence, using de Moivre's theorem

$$(1-i)^4=4\mathrm{e}^{-i\pi}=-4.$$

23 ADDITIONAL PROBLEMS AND COMMENTS

In this section we work out some additional problems which make use of de Moivre's theorem. We also show how n-th root of unity helps us determine the roots of a complex number. We also give a brief description of *cyclo-tomic* equations and its role in finding roots of a complex number.

Example 1. Find the phase angle θ when $z = (\sqrt{3} - i)^6$.

The simplest way to find the phase angle θ is to make use of polar representation of the complex number. Thus

$$z = (\sqrt{3} - i)^6 = (\sqrt{3 + 1}\mathrm{e}^{i\theta})^6 = (2\mathrm{e}^{i\theta})^6,$$

where $\theta = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6} = (11/6)\pi$. Hence, the phase angle is $\theta = (11/6)\pi$, measured positive counter-clockwise.

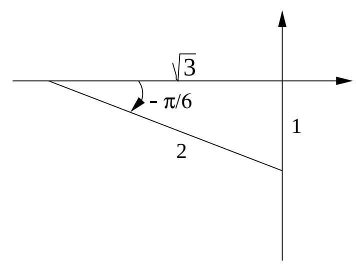

Figure 19.

Example 2. By writing individual factors on the left side of the equation in exponential form, show that

$$i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i).$$

The exponential representation of each one of the three simple factors is

1. $i = \mathrm{e}^{i\pi/2}$;
2. $(1 - \sqrt{3}i) = \sqrt{1+3}\exp[i(-\tan^{-1}\sqrt{3})] = 2\mathrm{e}^{-i\pi/3}$;
3. $(i + \sqrt{3}) = \sqrt{3+1}\exp(i\tan^{-1}\frac{1}{\sqrt{3}}) = 2\mathrm{e}^{i\pi/6}$.

Hence, taking the product of these three factors we find

$$\begin{aligned} i(1 - \sqrt{3}i)(\sqrt{3} + i) &= 1 \times 2 \times 2 \times \exp[i\pi(\frac{1}{2} - \frac{1}{3} + \frac{1}{6})], \\ &= 4\exp[i\frac{\pi}{6}(3 - 2 + 1)] = 4\mathrm{e}^{i\pi/3}, \\ &= 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2(1 + \sqrt{3}i). \qquad \diamond \end{aligned}$$

Example 3. Using polar representation show that

$$(-1 + i)^7 = -8(1 + i).$$

Using polar representation we write

$$(-1+i) = \sqrt{2}\mathrm{e}^{i3\pi/4}.$$

Therefore

$$\begin{aligned}
(-1+i)^7 &= 2^{7/2}\mathrm{e}^{i\frac{21}{4}\pi}, \\
&= 8\sqrt{2}\mathrm{e}^{i(4\pi+5\pi/4)}, \\
&= 8\sqrt{2}\mathrm{e}^{i5\pi/4}, \\
&= -8\sqrt{2}\mathrm{e}^{i\pi/4} = -8(1+i). \qquad \diamond
\end{aligned}$$

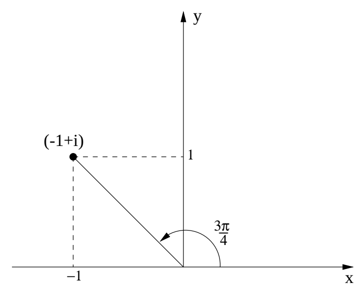

Figure 20.

24 PRINCIPAL ARGUMENT

Consider a complex number $z = (1-i)$. This complex number lies in the fourth quadrant and its polar representation is

$$z = (1-i) = \sqrt{2}\mathrm{e}^{-i\pi/4}.$$

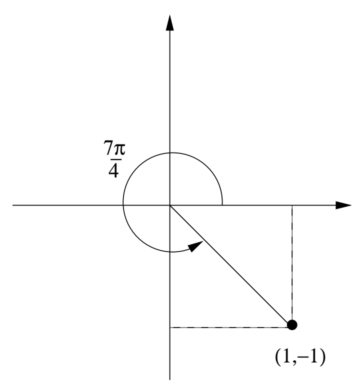

Figure 21.

However, in polar representation the argument of the function is not unique because

$$\begin{aligned}
z &= \sqrt{2}\mathrm{e}^{-i\pi/4}, \\
&= \sqrt{2}\mathrm{e}^{i(-\pi/4+2k\pi)}, \quad k = 0, \pm1, \pm2, \cdots, \pm n, \cdots \\
&= \sqrt{2}\mathrm{e}^{i(\Theta+2k\pi)}, \quad k = 0, \pm1, \pm2, \cdots, \pm n, \cdots;
\end{aligned}$$

where $\Theta = \pi/4$ is called the *principal* argument and we write it as

$$\text{Arg}z = \Theta.$$

In terms of the principal argument, the argument of the complex number z is defined as

$$\text{arg}z = \Theta + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots, \pm n, \cdots.$$

We note that arg$z$, namely argument of the complex variable z differs from the principal argument (Arg$z$) by real values differing by integral values of 2π. Hence, in general we write

$$\text{arg}z = \text{Arg}z + 2k\pi, \quad k = 0, \pm 1, \pm 2, \cdots, \pm n, \cdots$$

or

$$\theta = \Theta + 2k\pi. \qquad \diamond$$

In our present problem the principal argument is $\Theta = -\frac{\pi}{4}$ and arg$z = \theta = -\frac{\pi}{4} + 2k\pi$, where $k \in \mathcal{Z}$, the set of all real integers.

25 THE n ROOTS OF UNITY

The n roots of a complex number w can be easily determined if we know the n roots of unity. Thus if w is a complex number, then the n roots of the equation

$$z^n = w,$$

can be easily determined if we know the n roots of unity (when $w = 1$). Using the polar form of representation we write

$$z^n = r^n e^{in\theta} = 1 = e^{2ik\pi}.$$

Equating the modulus and the phase, we find

$$\begin{aligned} r^n &= 1, \\ n\theta &= 2k\pi \text{ for some } \quad k \in \mathcal{Z}. \end{aligned}$$

Obviously $r = 1$, since modulus is positive. And, the allowable values of θ are

$$\theta = 2\pi(k/n), \quad k = 0, \pm 1, \pm 2, \cdots, (n-1); \qquad n \geq 1.$$

Hence, the n distinct solutions are

$$\begin{aligned} z_0 &= e^{2i\pi(0/n)} = e^0 \equiv 1, \\ z_1 &= e^{2i\pi(1/n)} = e^{2i\frac{\pi}{n}} \equiv w, \\ z_2 &= e^{2i\pi(2/n)} = e^{2(2i\frac{\pi}{n})} \equiv w^2, \\ z_3 &= e^{2i\pi(3/n)} = e^{3(2i\frac{\pi}{n})} \equiv w^3, \\ &\cdots\cdots\cdots \\ z_{n-1} &= e^{2i\pi(\frac{n-1}{n})} = e^{(n-1)(2i\frac{\pi}{n})} \equiv w^{n-1}, \qquad n \geq 1. \end{aligned}$$

We first note that

$$z_{n-1} = \mathrm{e}^{2i\pi(\frac{n-1}{n})} = \mathrm{e}^{-2i\frac{\pi}{n}} = \frac{1}{w}.$$

However, we have also shown that

$$z_{n-1} \equiv w^{n-1}.$$

Therefore

$$z_{n-1} \equiv w^{n-1} = \frac{1}{w},$$

or w satisfies the equation

$$w^n = 1.$$

The invariants of the polynomial equation

$$z^n = 1,$$

are

$$\mathbf{I:}\ z_0 + z_1 + z_2 + \cdots + z_n = 1 + w + w^2 + \cdots + w^{n-1} \equiv 0.$$

$$\begin{aligned}\mathbf{II:}\ & z_0 z_1 + z_0 z_2 + z_0 z_3 + \cdots + z_1 z_2 + z_1 z_3 + \cdots + z_2 z_3 + \cdots \\ & = (w + w^2 + w^3 + w^4 + \cdots) + w(w^2 + w^3 + w^4 + \cdots) + w^2(w^3 + w^4 + \cdots) = 0. \\ & \quad \vdots\end{aligned}$$

The first invariant

$$\mathbf{I} = (1 + w + w^2 + w^3 + \cdots + w^{n-1}) = 0, \qquad n \geq 1.$$

is known as the *cyclo-tomic* equation (i.e., the circle-dividing equation). The argument $\theta = 2\pi(k/n)$ and therefore the roots of unity are equally spaced around the circle of radius $|z| = r = 1$. The figure below shows the position of the roots for $n = 6$.

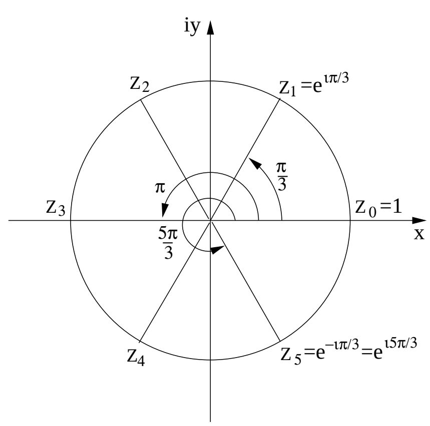

Figure 22.

26 THE n ROOTS OF A COMPLEX NUMBER

Let w be a complex number and let us assume that we have to find the n roots of w. Using polar representation we write

$$\begin{aligned} w &= |w|\mathrm{e}^{i\theta}, \\ &= |w|\mathrm{e}^{i(\Theta+2k\pi)}, \qquad k = 0, \pm 1, \pm 2, \cdots . \end{aligned}$$

Then the n roots of the complex number w are

$$z_k = |w|^{\frac{1}{n}} \mathrm{e}^{i(\frac{\Theta}{n}+\frac{2k\pi}{n})},$$

or

$$z_k = |w|^{\frac{1}{n}} \mathrm{e}^{i\frac{\Theta}{n}} w^k, \quad k = 0, \pm 1, \pm 2, \cdots, \pm(n-1),$$

where w is the n-th root of unity and as shown in §25, it is defined as

$$w = \mathrm{e}^{i\frac{2\pi}{n}} \qquad n \geq 1.$$

Hence, we can obtain n roots of complex number w by multiplying the principal root z_0 with powers of w, where $z_0 \equiv |w|^{\frac{1}{n}} \exp(i\Theta/n)$.

Example 1. Find the cube roots of the complex number

$$w = (-4\sqrt{2} + i4\sqrt{2})^{1/3}.$$

Using polar representation the principal root is

$$z_0 = (\sqrt{64}\mathrm{e}^{i\Theta})^{1/3} = (8\mathrm{e}^{i\Theta})^{1/3} = 2\mathrm{e}^{i\Theta/3},$$

where $\mathrm{Arg} z_0 \equiv \Theta = -\tan^{-1} 1 = \frac{3\pi}{4}$. Hence

$$\begin{aligned} z_0 &= 2\mathrm{e}^{i\pi/4}, \\ &= \sqrt{2}(1+i), \end{aligned}$$

is the principal root of the given complex number w.

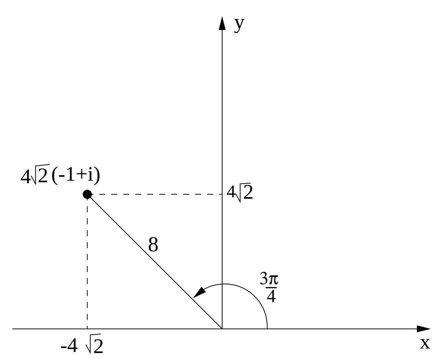

Figure 23.

The three cube roots of unity are

$$w^k = \mathrm{e}^{i\frac{2k\pi}{3}}, \qquad k = 0, 1, 2,$$

which gives us

$$
\begin{aligned}
w^0 &= 1, \\
w^1 &= \exp\left(i\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \\
&= \frac{1}{2}(-1 + i\sqrt{3}); \\
w^2 &= \exp\left(i\frac{4\pi}{3}\right) = -\exp\left(i\frac{\pi}{3}\right), \\
&= -(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = -\frac{1}{2}(1 + i\sqrt{3}).
\end{aligned}
$$

The three roots of the complex number w can now be readily obtained. These three roots are

$$z_0, \quad z_1 = z_0 w \quad , \text{and} \quad z_2 = z_1 w = z_0 w^2.$$

Thus

$$
\begin{aligned}
z_0 &= 2e^{i\pi/4} = \sqrt{2}(1+i), \\
z_1 &= 2\exp\left(i\frac{\pi}{4}\right)\exp\left(i\frac{2\pi}{3}\right) = \sqrt{2}(1+i)\frac{1}{2}(-1+i\sqrt{3}), \\
&= \frac{1}{\sqrt{2}}(1+i)(-1+i\sqrt{3}) = \frac{1}{\sqrt{2}}[-(\sqrt{3}+1) + i(\sqrt{3}-1); \\
z_2 &= 2\exp i\pi/4 \exp i4\pi/3 = \sqrt{2}(1+i)(-\frac{1}{2})(1+i\sqrt{3}), \\
&= -\frac{1}{\sqrt{2}}(1+i)(1+i\sqrt{3}) = \frac{1}{\sqrt{2}}[(\sqrt{3}-1) - i(\sqrt{3}+1).
\end{aligned}
$$

The three roots of w are shown in the figure below.

Also note that with these values of the roots z_0, z_1 andz_2, the three principal invariants { **I,II, III** } take the desired values $\{0, 0, -4\sqrt{2}(1-i)\}$.

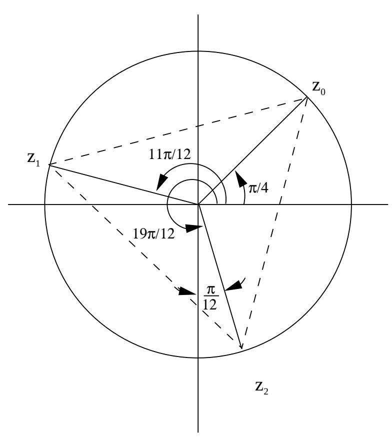

Figure 24.

Example 2. Find the cube roots of the complex number

$$w = (-8i)^{1/3}.$$

Using polar decomposition of a complex number, the principal root is

$$z_0 = (8e^{\frac{3i\pi}{2}})^{1/3} = 2e^{i\pi/2}.$$

The cube roots of unity are

$$w^k = e^{i\frac{2k\pi}{3}}, \quad k = 0, 1, 2.$$

Hence

$$\begin{aligned}
w^0 &= 1, \\
w^1 &= \exp i\frac{2\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \\
&= -\frac{1}{2}(1 - i\sqrt{3}); \\
w^2 &= \exp i\frac{4\pi}{3} = -\exp i\frac{\pi}{3}, \\
&= -(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = -\frac{1}{2}(1 + i\sqrt{3}).
\end{aligned}$$

We may note that w^0, w^1 and w^2 satisfy the cyclo-tomic equation

$$w^0 + w^1 + w^2 = 1 - \frac{1}{2}(1 - \sqrt{3}) - \frac{1}{2}(1 + \sqrt{3}) \equiv 0.$$

The cube roots of the complex numbers w are obtained from the principal root z_0 by multiplying it with w^0, w^1 and w^3. Thus the three roots are

$$\begin{aligned}
z_0 &= 2e^{i\pi/2}w^0 = 2i, \\
z_1 &= z_0 w^1 = 2i[-\frac{1}{2}(1 - i\sqrt{3})] = -(\sqrt{3} + i), \\
z_2 &= z_0 w^2 = 2i[-\frac{1}{2}(1 + i\sqrt{3})] = (\sqrt{3} - i).
\end{aligned}$$

We now show that the three principal invariants take the desired values:

$$\begin{aligned}
\mathbf{I:}\ & z_0 + z_1 + z_2 = 2i - (\sqrt{3} + i) + (\sqrt{3} - i) \equiv 0. \\
\mathbf{II:}\ & z_0 z_1 + z_0 z_2 + z_1 z_2 = (-2i\sqrt{3} + 2) + (2i\sqrt{3} + 2) - 4 \equiv 0. \\
\mathbf{III:}\ & z_0 z_1 z_2 = -2i(\sqrt{3} + i)(\sqrt{3} - i) = -2i \times 4 \equiv -8i.
\end{aligned}$$

◇

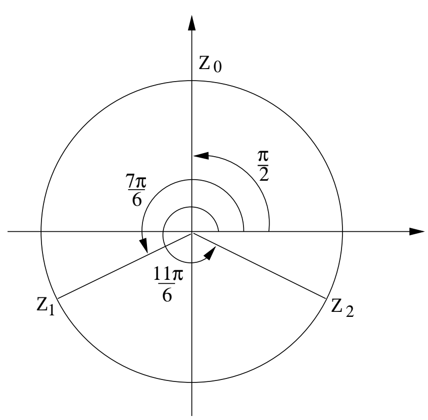

Figure 25.

27 COMPLEX NUMBERS AND THE VALUE OF π

De Moivre's theorem for complex numbers can be used to evaluate the transcedental number π. At first sight it may seem to be an impossibility, but that is the power of de Moivre's theorem. We consider the following examples:

Example 1. Evaluate the complex number

$$z = (2+i)(3+i),$$

and show that

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right).$$

Using polar decomposition we write

$$\begin{aligned}(2+i) &= \sqrt{5}\exp[i\tan^{-1}(1/2)],\\ (3+i) &= \sqrt{10}\exp[i\tan^{-1}(1/3)].\end{aligned}$$

Therefore, the product of the two factors is

$$(2+i)(3+i) = \sqrt{50}\exp\{i[\tan^{-1}(1/2) + \tan^{-1}(1/3)]\}.$$

However

$$\begin{aligned}\tan^{-1}(1/2) + \tan^{-1}(1/3) &= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right),\\ &= \tan^{-1}\frac{5/6}{1-1/6} = \tan^{-1}1 = \frac{\pi}{4}.\end{aligned}$$

Hence

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right). \qquad \bullet$$

EXERCISE. Evaluate the complex number $z = (3+2i)(5+i)$ and show that:

$$\frac{\pi}{4} = \tan^{-1}(1/5) + \tan^{-1}(2/3). \qquad \bullet$$

EXERCISE. Show that:

$$\frac{\pi}{4} = 2\tan^{-1}(1/3) + \tan^{-1}(1/7). \qquad \bullet$$

EXERCISE. Show that:

$$\frac{\pi}{4} = 4\tan^{-1}(1/5) - \tan^{-1}(1/239). \qquad \bullet$$

Example 2. Evaluate the complex number

$$z = (5 - i)^4 (1 + i),$$

and show that

$$\frac{\pi}{16} = \tan^{-1}\left(\frac{1}{5}\right) - \frac{1}{4}\tan^{-1}\left(\frac{1}{239}\right).$$

This is called Machin's formula and has been used to compute the value of π to a high degree of accuracy. Machin himself computed the value of π to 100 decimal places, and in 1717 DeLagny of France computed the value of π to 127 decimal places. We now show the derivation.

Using polar decomposition, we write

$$\begin{aligned}(5 - i)^4 (1 + i) &= 676\sqrt{2}\{\exp[i(-\tan^{-1}\frac{1}{5} + \tan^{-1} 1)]\}, \\ &= 676\sqrt{2}\exp[i(\frac{\pi}{4} - \tan^{-1}\frac{1}{5})].\end{aligned}$$

Let $\tan\beta = 1/5$. Then $\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2/5}{1-1/25} = 5/12$, and $\tan 4\beta = \frac{2\tan 2\beta}{1-\tan^2 2\beta} = \frac{5/6}{1-25/144} = 120/119$. Now $\tan 4\beta = 1 + 1/119 = \tan\frac{\pi}{4} + 1/119$, and therefore $(\tan 4\beta - \tan\frac{\pi}{4}) = 1/119$. Similarly it follows that $(\tan 4\beta + \tan\frac{\pi}{4}) = 239/119$. Consequently the quotient is

$$\frac{\tan 4\beta - \tan\frac{\pi}{4}}{\tan 4\beta + \tan\frac{\pi}{4}} = \tan(4\beta - \frac{\pi}{4}),$$

and therefore

$$\tan(4\beta - \frac{\pi}{4}) = \frac{1}{119} \times \frac{119}{239} = \frac{1}{239}.$$

From here we obtain

$$4\beta - \frac{\pi}{4} = \tan^{-1}\frac{1}{239},$$

or

$$\frac{\pi}{16} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

We thus have Machin's formula

$$\frac{\pi}{16} = \tan^{-1}\frac{1}{5} - \frac{1}{4}\tan^{-1}\frac{1}{239}. \qquad \bullet$$

Machin calculated the value of π by replacing the arctangent functions by Gregory series. He thus obtained

$$\begin{aligned}\frac{\pi}{16} &= \left(\frac{1}{5} - \frac{1}{3} \times \frac{1}{5^3} + \frac{1}{5} \times \frac{1}{5^5} - \cdots\right) \\ &\quad - \frac{1}{4}\left(\frac{1}{239} - \frac{1}{3} \times \frac{1}{239^3} + \frac{1}{5} \times \frac{1}{239^5} - \cdots\right). \qquad \blacksquare\end{aligned}$$

The second series converges very rapidly and the first one is well suited for numerical calculations, because successive terms in the series diminish by a factor of $(1/5^2) = 0.04$. From this formula John Machin, Professor of Astronomy in London, calculated in 1706 the value of π correct to 100 decimal values. It was at about this time the symbol π was first introduced by William Jones (1675-1749), who used the **greek** symbol for the english word *"periphery"* for a circle of unit diameter. However, it became a standard symbol only after Euler adopted it, as has been the case with many other symbols which have entered in the mathematical literature.

We may note in passing that modern calculations to over two billion digits are based on far more complicated series such as the one published by S. Ramanujan (1887–1920) in 1915:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{(1103+26390n)}{396^{4n}}. \qquad \blacksquare$$

Each additional term of this series roughly adds 8 digits to the value of π.

Another useful series due to Ramanujan is: This ia a margin-al note

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\binom{2n}{n}^3\left(\frac{42n+5}{2^{(12n+4)}}\right). \qquad \blacksquare$$

EXERCISE . Show that, [S. L. Loney (1893), C. Strőmer (1896), R. W. Morris (1944)]:

$$\frac{\pi}{4} = 3\tan^{-1}(1/4) + \tan^{-1}(1/20) + \tan^{-1}(1/1985). \qquad \bullet$$

EXERCISE. Show that:

$$\frac{\pi}{4} = 8\tan^{-1}(1/10) - \tan^{-1}(1/239) - 4\tan^{-1}(1/515). \qquad \bullet$$

EXERCISE. Show that:

$$\frac{\pi}{4} = 12\tan^{-1}(1/18) + 8\tan^{-1}(1/57) - 5\tan^{-1}(1/239). \qquad \bullet$$

EXERCISE. Show that:

$$\frac{\pi}{4} = 6\tan^{-1}(1/8) + 2\tan^{-1}(1/57) + \tan^{-1}(1/239). \qquad \bullet$$

28 EQUALITY OF TWO COMPLEX NUMBERS

Consider two complex numbers

$$z_1 = x_1 + iy_1,$$
$$z_2 = x_2 + iy_2.$$

If these two complex numbers are equal, then

$$x_1 + iy_1 = x_2 + iy_2,$$

or

$$(x_1 - x_2) + i(y_1 - y_2) = 0.$$

Multiplying by its conjugate, we get

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 0,$$

For this to be true for all values of z_1 and z_2, we must have

$$x_1 = x_2 \text{ and } y_1 = y_2,$$

i.e., if two complex numbers are equal, then their real and imaginary parts must be equal.

If we write these complex numbers in polar form, then

$$z_1 = r_1 \mathrm{e}^{i\Theta_1},$$
$$z_2 = r_2 \mathrm{e}^{i\Theta_2},$$

where Θ_1 and Θ_2 are the principal arguments of z_1 and z_2. However

$$\Theta_1 = \Theta_2 + \mathrm{mod}(2\pi).$$

Therefore in the case of equality, we must have

$$r_1 = r_2,$$
$$\Theta_1 = \Theta_2 + \mathrm{mod}(2\pi).$$

From the first equation we find

$$x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

The points (x_1, y_1) and (x_2, y_2) are arbitrary. Therefore, for this equation to be true for all points, it must be true for all points on the x-axis, and also for all points on the y-axis. Equality of two complex numbers therefore requires that $x_1 = x_2$ and $y_1 = y_2$. The condition for the equality of arguments is then satisfied trivially.

29 APPENDIX A : TRIGONOMETRIC IDENTITIES

In this section we establish some useful trigonometric identities by using simple geometric considerations. In the second part of this section we show how these identities can be obtained much more simply by using Euler's exponential identity and de Moivre's theorem in the complex number plane.

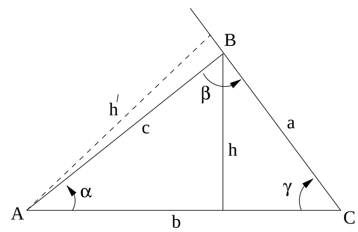

Figure 26.

We first derive the *law of sines* and *law of cosines* for any plane triangle, because we will need these results.

LAW OF SINES:
Let A, B and C be the vertices and a, b and c be the lengths of the opposite sides of the triangle ABC. The angles α, β, γ are the interior angles of the triangle at the vertices A, B and C. The sum of the three interior angles α, β, γ of a plane triangle is a straight angle or 180^o in degree measure or π radians in circular measure, i.e.,

$$\alpha + \beta + \gamma = \pi \text{ (in radians)}.$$

Let h be the height of the point B from the base AC. Then, from the definition of the sine function, we find that

$$h = a \sin\gamma = c \sin\alpha.$$

We thus obtain

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}. \qquad \diamond$$

Next let us assume that the normal distance from the vertex A to the side BC is h'. Then

$$\begin{aligned} h' &= b\sin\gamma = c\sin(\pi - \beta), \\ &= c(\sin\pi\cos\beta - \cos\pi\sin\beta), \\ &= c\sin\beta. \end{aligned}$$

Therefore we find

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}. \qquad \diamond$$

Combining the two identities, we finally obtain

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}, \qquad \blacksquare$$

which is called the law of sines.

LAW OF COSINES:

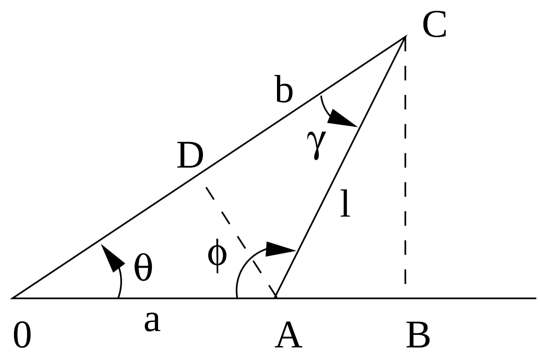

Figure 27.

Using the distance formula

1. $$l^2 = (AB)^2 + (BC)^2 = (OB - OA)^2 + (BC)^2,$$
$$= (b\cos\theta - a)^2 + (b\sin\theta)^2 = a^2 + b^2 - 2ab\cos\theta. \qquad \diamond$$

2. $$b^2 = (OB)^2 + (BC)^2 = (OA + AB)^2 + [l\sin(\pi - \phi)]^2,$$
$$= [a + l\cos(\pi - \phi)]^2 + [l\sin(\pi - \phi)]^2 = a^2 + l^2 + 2al\cos(\pi - \phi),$$
$$= a^2 + l^2 - 2al\cos\phi. \qquad \diamond$$

3. $$a^2 = (AD)^2 + (OD)^2 = (l\sin\gamma)^2 + (OC - CD)^2,$$
$$= (l\sin\gamma)^2 + (b - l\cos\gamma)^2 = l^2 + b^2 - 2bl\cos\gamma. \qquad \diamond$$

In the formulas above $\theta + \phi + \gamma = 2\pi$ in radian measure. These three formulas are called the *law of cosines.* If we add the three results we find

$$l^2 + a^2 + b^2 = 2(l^2 + a^2 + b^2) - 2(ab\cos\theta + al\cos\phi + bl\cos\gamma),$$

or, cancelling similar terms from the two sides of this equation we find that for a plane triangle

$$(l^2 + a^2 + b^2) = 2(ab\cos\theta + al\cos\phi + bl\cos\gamma).$$

Let ϕ be a right angle, so that $\cos\phi = 0$. Also, in the case of a right-angle triangle we have the Pythagorean formula for distance $b^2 = a^2 + l^2$. Hence when $\phi = \pi/2$

$$l^2 + a^2 + b^2 = 2(l^2 + a^2) = 2(ab\cos\theta + bl\cos\gamma),$$

or

$$l^2 + a^2 = ab\cos\theta + bl\cos\gamma, \qquad \theta + \gamma = \pi/2\,,$$
$$= b(a\cos\theta + l\cos\gamma).$$

However, it is geometrically obvious that

$$a\cos\theta + l\cos\gamma = b,$$

and therefore, for a right-angled triangle

$$l^2 + a^2 = b^2.$$

TRIGONOMETRICAL IDENTITIES

We are now in a position to establish the following two sets of trigonometrical **identities**:

$$1. \qquad \begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta. \end{aligned} \qquad \blacksquare$$

$$2. \qquad \begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta) = \cos(\alpha+\beta)], \\ \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)], \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)], \\ \cos\alpha\sin\beta &= \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]. \end{aligned} \qquad \blacksquare$$

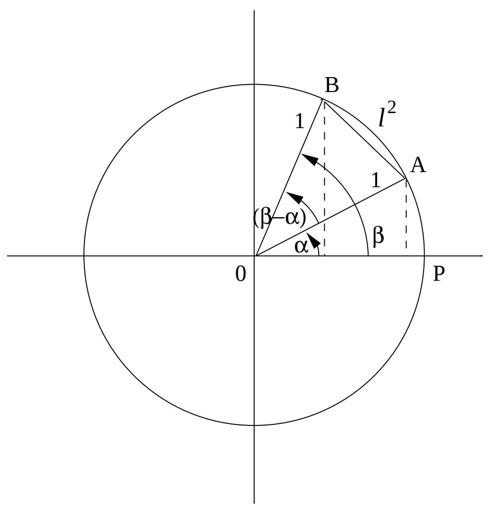

Figure 28.

PROOF : Consider a circle of unit radius. Let $\angle POB = \beta$ and $\angle POA = \alpha$, in radians. Then $\angle AOB = (\beta - \alpha)$.

The coordinates of the point **A** are $(\cos\alpha, \sin\alpha)$ and the coordinates of the point **B** are $(\cos\beta, \sin\beta)$. Using the distance formula, the length of the chord AB is:

$$\begin{aligned} |AB|^2 = l^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\beta - \sin\alpha)^2, \\ &= (\cos^2\beta + \cos^2\alpha - 2\cos\beta\cos\alpha) + (\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta), \\ &= (\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta), \\ &= 2[1 - (\cos\alpha\cos\beta + \sin\alpha\sin\beta)]. \end{aligned}$$

However, using the law of cosines, when OA=OB=1, we find

$$l^2 = 2[1 - \cos(\alpha - \beta)],$$

since $a = b = 1$. Hence, equating the two representations for l^2, we find the first trigonometric identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

If in this identity we put $\beta \to -\beta, \quad \cos(-\beta) = \cos\beta, \quad \sin(-\beta) = -\sin\beta$, we get the second identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Next replace $\beta \to \beta + \frac{\pi}{2}$. Then

$$\begin{aligned}
\cos(\alpha - \beta - \frac{\pi}{2}) &= \cos\alpha\cos(\beta + \frac{\pi}{2}) + \sin\alpha\sin(\beta + \frac{\pi}{2}), \\
\cos(\alpha + \beta + \frac{\pi}{2}) &= \cos\alpha\cos(\beta + \frac{\pi}{2}) - \sin\alpha\sin(\beta + \frac{\pi}{2}).
\end{aligned}$$

Now

$$\begin{aligned}
\sin(\beta + \frac{\pi}{2}) &= \cos\beta, \\
\cos(\beta + \frac{\pi}{2}) &= -\sin\beta, \\
\cos(\alpha - \beta - \frac{\pi}{2}) &= \sin(\alpha - \beta), \\
\cos(\alpha + \beta + \frac{\pi}{2}) &= -\sin(\alpha + \beta).
\end{aligned}$$

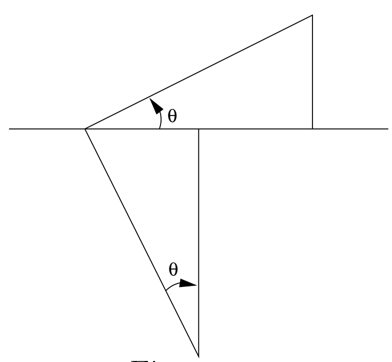

Figure 29.

Therefore we have the second set of identities

$$\begin{aligned}
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}$$

These identities are true for all values of α and β. In particular when $\alpha = \beta$, we find the Pythagorean identity

$$\cos^2\alpha + \sin^2\alpha = 1.$$

We also obtain the identity

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha.$$

Combining these two, we find

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha),$$
$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha).$$

Also when $\alpha = \beta$, from the identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

we obtain

$$\sin 2\alpha = 2\sin\alpha\cos\alpha.$$

We may note in passing that the second set of four identities can be obtained from the first set of four by suitable addition and/or subtraction.

DOUBLE ANGLES AND HALF ANGLES

The double-angle formulas obtained in the last section were

$$\sin 2\alpha = 2\sin\alpha\cos\alpha,$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$$

From these two formulas we obtain

$$\tan 2\alpha = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}, \quad = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

To derive half-angle formulas, we put $2\alpha = \theta$ in these equations. Then

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$
$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},$$
$$\tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2}.$$

Also

$$1+\cos\theta = (1-\sin^2\frac{\theta}{2}) + \cos^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2},$$
$$1-\cos\theta = (1-\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2} = 2\sin^2\frac{\theta}{2}.$$

Therefore, dividing the two we find

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad , \text{ if } \tan\frac{\theta}{2} \text{ is } \begin{cases} \text{positive,} \\ \text{negative.} \end{cases}$$

30 APPENDIX B : EULER'S CONSTANT e

In differentiating logarithmic functions there appears a quantity $\lim\limits_{h\to 0}(1+h)^{1/n}$, which can be shown to approach a finite limit as $h \to 0$. We now write $h = 1/n$ and examine the limit

$$\lim_{n\to\infty}(1+\frac{1}{n})^n \text{ for positive integral values of } n.$$

Using binomial expansion this can be written as

$$\begin{aligned}
(1+\frac{1}{n})^n &= 1 + \binom{n}{1}(\frac{1}{n}) + \binom{n}{2}(\frac{1}{n})^2 + \binom{n}{3}(\frac{1}{n})^3 + \cdots + \binom{n}{n}(\frac{1}{n})^n, \\
&= 1 + n(\frac{1}{n}) + \frac{n(n-1)}{1\times 2}(\frac{1}{n})^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(\frac{1}{n})^3 + \cdots + (\frac{1}{n})^n, \\
&= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + (\frac{1}{n})^n.
\end{aligned}$$

The last term may be written as

$$\begin{aligned}
(\frac{1}{n})^n &= \frac{n(n-1)(n-2)\cdots 3\times 2\times 1}{n!}(\frac{1}{n})^n, \\
&= \frac{1}{n!}\,\frac{n(n-1)(n-2)\cdots 3\times 2\times 1}{n^n}, \\
&= \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\cdots(1-\frac{n-1}{n}).
\end{aligned}$$

Hence, putting these three terms together, we get

$$\begin{aligned}
(1+\frac{1}{n})^n &= 2 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots \\
&+ \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{k}{n}) + \cdots \\
&+ \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\cdots(1-\frac{n-1}{n}).
\end{aligned}$$

As $n \to \infty$, the right-hand side of this equation becomes an infinite series whose $(k+1)$-st term is $1/k!$. Thus in the limit we find

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!} + \cdots .$$

We now show that this infinite sum converges. To prove that this infinite series converges it is sufficient to show that it is bounded from above. Thus

$$\begin{aligned}
\lim_{n\to\infty}(1+\frac{1}{n})^n &= 2 + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \frac{1}{1\times 2\times 3\times 4} + \cdots + \frac{1}{1\times 2\times 3\times\cdots\times k} + \cdots , \\
&= 2 + \frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{2\times 3\times 4} + \cdots + \frac{1}{2\times 3\times\cdots\times k} + \cdots , \\
&< 2 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \cdots + (\frac{1}{2})^{n-1} + \cdots , \\
&= 1 + \frac{1-(1/2)^n}{1-1/2}, \\
&= 1 + 2[1-(\frac{1}{2})^n], \\
&= 3 - (\frac{1}{2})^{n-1} < 3 \text{ as } n \to \infty.
\end{aligned}$$

Hence, the infinite series is bounded from above. In addition the sum of the series increases monotonically. A monotonically increasing series bounded from above converges and therefore $\lim_{n\to\infty}(1+\frac{1}{n})^n$ converges and we call it Euler's constant e [9]. Thus

$$\mathrm{e} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!} + \cdots . \qquad \text{Q.E.D.}$$

This infinite series [10] can be easily tabulated and its value is

$$\mathrm{e} = 2.7\ 1828\ 1828\ 459045 \cdots . \qquad \blacksquare$$

In fact, Euler calculated the value to the twenty-fifth decimal position.

[9] Leonhard Euler [1707-1783] (pronounced "Oiler"), the great Swiss mathematician, from Basel (Switzerland) introduced the number e and also named it in his calculus text *"Introductio in analisin infinitorum"*, vol.1, (1748), p.90.

[10] Quod Erat Demonstrandum *(dA–mon–străn–dum)*, abbreviated by *Q.E.D.* is Latin for : *which was to be proved,* and thus signifies that the proof has been concluded. It is a Latin translation of the phrase $o\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \delta\epsilon\iota\zeta\alpha\iota$, and Euclid concluded most of his proofs with this acronym.

31 APPENDIX C : THE FASCINATING STORY OF i

The quadratic formula for the solution of a second order polynomial equation is well known for the last 4000 years. It was originally believed that when the discriminant of the quadratic equation is negative, it has no solution. However, the discovery of **Gerolama Cardan's** formula (1545) for a cubic equation presented serious problems. Early in the sixteenth century it was first shown by **Scipione del Ferrro** (1464–1526) of Pisa, and then by **Niccolo Fontana** (1499–1557) that every cubic equation can be put in a depressed form [11]

$$x^3 + 3px = 2q,$$

and he showed that the general solution is

$$x = \sqrt[3]{q + \sqrt{\Delta}} + \sqrt[3]{q - \sqrt{\Delta}},$$

where the *discriminant* of the depressed equation is:

$$\Delta \equiv \left(q^2 + p^3\right).$$

The solution in this form is known as *Cardan's formula.*

This form of general solution presented an immediate problem for solutions of the equation

$$x^3 - 15x = 4, \tag{92}$$

because in this case the discriminant is an imaginary number $\Delta \equiv 11i$. [12]

Using Cardan's formula the solution is

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}. \tag{93}$$

However, as can be easily verified, the three real-valued solutions of the depressed equation are

$$\begin{aligned} x &= 4, \\ x &= -(2 \pm \sqrt{3}). \end{aligned} \tag{94}$$

To show that Cardan's solution (93) is equivalent to the three solutions (94) took another two centuries of intensive work. It was only after **Euler** introduced the imaginary number $i = \sqrt{-1}$ [13] in calculus, it was possible to show that Cardan's formula can be rewritten as:

$$x = (2 + 11i)^{1/3} + (2 - 11i)^{1/3}. \tag{95}$$

Finally, with the help of **DeMoivre's** theorem it can be shown that the complex number has indeed three roots and when the solution is explicitly of the form (95), the three solutions are real.

[11] In the history of mathematics, Fontana is known as **Tartaglia** ("stammerer"). Cardano learned the secret of the depressed cubic equation from Tartaglia, and he showed that a general cubic equation $y^3 + py^2 + qy + r = 0$ can be reduced to the depressed form by using the transformation $y = x - p/3$.

[12] Such a problem does not arise if the discriminant of the depressed cubic equation is a real number. As an example the discriminant of the equation $x^3 + 6x - 20 = 0$ is $\Delta \equiv \sqrt{108}$, which is a real number. In this case Cardan's formula yields $x = 2$, as one of the solution. The depressed cubic can therefore be written in the factored form $(x - 2)(x^2 + 2x + 20)$. From here we easily find that the three solutions of the cubic equation are $x = 2; (-1 + i\sqrt{19}); (-1 - i\sqrt{19})$.

[13] An Egyptian papyrus found in the tombs of Pharaoh's in the Valley of the Kings contains the first account of the square root of a negative number. **Heron** of Alexandria also encountered the imaginary number i.

This success led to important research by Gauss and Cauchy in the theory of complex variables — they showed its importance in contour integration, residue theory and Fourier transforms [14]. Without this imaginary number i, all these developments would have been well-nigh impossible. But then electrical engineering also would have been impossible because it has been said that Steinmetz of Niagara Falls, NY, (1865-1923) [15], "generated electricity out of the imaginary number i" (which of course he labelled as j).

[14] The "imaginary number" i, also called "the notorious, or elusive number" was finally accepted and put to use in Napoleonic times by Cauchy and Fourier.

[15] Steinmetz, Charles Proteus, was an influential electrical engineer in General Electric, who was responsible for the electrification of Buffalo, NY. His statue can be seen in Niagara Falls, NY.

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